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An efficient third-moment saddlepoint approximation for probabilistic uncertainty analysis and reliability evaluation of structures

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ABSTRACT

This paper presents an efficient third-moment saddlepoint approximation approach for probabilistic uncertainty analysis and reliability evaluation of random structures. By constructing a concise cumulant generating function (CGF) for the state variable according to its first three statistical moments, approximate probability density function and cumulative distribution function of the state variable, which may possess any types of distribution, are obtained analytically by using saddlepoint approximation technique. A convenient generalized procedure for structural reliability analysis is then presented. In the procedure, the simplicity of general moment matching method and the accuracy of saddlepoint approximation technique are integrated effectively. The main difference of the presented method from existing moment methods is that the presented method may provide more detailed information about the distribution of the state variable. The main difference of the presented method from existing saddlepoint approximation techniques is that it does not strictly require the existence of the CGFs of input random variables. With the advantages, the presented method is more convenient and can be used for reliability evaluation of uncertain structures where the concrete probability distributions of input random variables are known or unknown. It is illustrated and examined by five representative examples that the presented method is effective and feasible.

1. Introduction

Uncertainty is inherent and unavoidable in almost all engineering systems. Reliability analysis is an important tool for quantifying and dealing with all sorts of uncertainties. In structural reliability analysis, a limit-state function (also known as performance function or failure function) $Z = g(\boldsymbol{X})$ is usually defined by the performance or the failure mode of the structure. If the joint probability density function (PDF) $f_X(\boldsymbol{x})$ of the basic random variables is known, then the probability of failure can be evaluated with the integral, $P_f = P\{g(\boldsymbol{X}) < 0\} = \int_{\Omega_f} f_X(\boldsymbol{x})d\boldsymbol{x}$, where $\Omega_f = \{\boldsymbol{x}|g(\boldsymbol{x}) < 0\}$ denotes the failure domain in the space of basic random variables. Nevertheless, since $Z = g(\boldsymbol{X})$ is usually a nonlinear function of numerous random variables, it is usually very difficult or even impossible to find an analytical expression to the multidimensional integration except for some extremely special cases. This difficulty has motivated the development of various approximate methods for structural reliability in the past decades. Traditionally, approximate analytical methods such as the well known first- and second-order reliability methods (FORM/SORM) [1–4], moment matching methods [5–10], and simulation methods [11–14] are often used to compute the failure probability. As is known, sampling simulation methods may be accurate enough

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if a sufficient number of simulations are used. However, when the function $Z = g(\boldsymbol{X})$ is complicated or for issues with high reliability, an excessively large number of simulations may be required and the sampling procedures may become impractical. In such cases, approximate analytical reliability method may be a reasonable alternative. It has been shown that FORM and SORM are accurate sufficiently if the design point is found out accurately. Generally, FORM is considered to be the most reliable computational method. But, in spite of its simplicity and usefulness, FORM is not accurate enough in many cases. SORM is more accurate than FORM but needs more computation. Furthermore, the existence of multiple design points on the limit state surface may cause gross error in common FORM/SORM [1]. Moreover, it is not feasible to carry out the multidimensional integration, whether analytically or numerically, if the joint PDF of the basic random variables is unknown.

Considering the fact that Z is a function of the basic random variables, it is a random variable too and is referred to as the state variable in the paper. The failure probability $P_f = P\{Z < 0\}$ can be obtained conveniently if the PDF of the state variable Z is known. However, because the state variable Z is usually a nonlinear function of the basic variables, its distribution is dependent not only on the actual form of the limit-state function $g(\cdot)$ but also on the distribution types of all the basic random variables. Except for some extremely special cases, it is usually difficult to obtain the accurate probability distribution of the state variable. Various methods have been developed to determine the approximate probability distribution of a distribution-unknown random variable and of a function of random variables (see for example [5–7,15–24]). In these methods, statistical moments matching method, which only requires finite information about statistical moments and calculates the reliability by using the statistical moments of limit-state function and fitting the moments with some empirical distribution systems [5–10], is a simple and effective way for reliability calculation. As a powerful tool for obtaining the PDF and cumulative distribution function (CDF) from associated cumulant (or moment) generating function, saddlepoint approximation (SA) is another effective technique. SA is an accurate method for estimating the CDF of a random variable if its cumulant generating function (CGF) is known [17]. It has played an increasingly important role since it was introduced into statistics [15–24]. Nevertheless, it requires that the variables involved are tractable, i.e., their CGFs exist. In the situations that the variables involved are intractable (the CGFs do not exist), the common SA techniques may be unusable. Moreover, the so-called saddlepoint equation must be solved to obtain the saddlepoint when the SA techniques are used. This is not always an easy job because the CGFs corresponding to most probability distribution types are complicated and so the saddlepoint equation may be highly nonlinear and for this reason may be difficult to solve.

Motivated by these considerations, the main purpose of this paper is to present an efficient method for probabilistic uncertainty analysis and reliability evaluation of random structures, in which the simplicity of moment matching method and the accuracy of SA techniques are integrated and utilized effectively. Because it does not require the existence of the CGFs of basic variables, the presented method may possess good applicability and can be carried out conveniently. The paper is organized as follows. In Section 2, the saddlepoint approximation techniques are briefly reviewed to provide preliminary knowledge. An efficient third-moment saddlepoint approximation approach is developed in detail in Section 3. In Section 4, five numerical examples are used to demonstrate and examine the presented methods. Conclusions are given in Section 5.

2. Preliminary knowledge – saddlepoint approximation

Saddlepoint approximation is a well-established technique for approximating the PDF and CDF of a random variable whose cumulant generating function (CGF) is known [20]. Assume that X is a random variable with the PDF $f_X(x)$, and its moment generating function (MGF), denoted by $M_X(\xi)$, exists. By the definition of MGF, there exists the relation

$$M_X(\xi) = \int_{\Omega_X} e^{\xi x} f_X(x)dx. \tag{1}$$

The corresponding CGF of X, denoted by $K_X(\xi)$, is related to the MGF $M_X(\xi)$ by the following natural logarithm operation

$$K_X(\xi) = \ln\{M_X(\xi)\}. \tag{2}$$

For continuous random variables, the saddlepoint approximation to the density of X

$$f_X(x) = \left\{2\pi K''_X(\hat{\xi})\right\}^{-1/2} \exp\left\{K_X(\hat{\xi}) - \hat{\xi}x\right\}, \tag{3}$$

was first introduced by Daniels [19]. In which, $\hat{\xi}$ is called the saddlepoint and is the solution to the saddlepoint equation

$$K'_X(\xi) = x, \tag{4}$$

where $K'_X(\cdot)$ and $K''_X(\cdot)$ are the first and second derivatives of the CGF $K_X(\xi)$. The most commonly used saddlepoint approximation, due to Lugannani and Rice [15,22], is given by

$$F_X(x) = P\{X \leqslant x\} = \Phi(r) + \phi(r)\left(\frac{1}{r} - \frac{1}{q}\right), \tag{5}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the standard normal CDF and PDF, respectively, with

$$r = sign(\hat{\xi})\left\{2[\hat{\xi}x - K_X(\hat{\xi})]\right\}^{1/2}, q = \hat{\xi}[K''_X(\hat{\xi})]^{1/2}, \tag{6}$$