Applied Mathematical Modelling 38 (2014) 253–262

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Nonlinear bending and buckling for strain gradient elastic beams

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ARTICLE INFO

Article history:
Received 29 September 2012
Received in revised form 18 May 2013
Accepted 24 June 2013
Available online 4 July 2013

Keywords:
Nonlinear bending
Strain gradient elastic
Thin beams
Surface energy

ABSTRACT

Nonlinear bending of strain gradient elastic thin beams is studied adopting Bernoulli–Euler principle. Simple nonlinear strain gradient elastic theory with surface energy is employed. In fact linear constitutive relations for strain gradient elastic theory with nonlinear strains are adopted. The governing beam equations with its boundary conditions are derived through a variational method. New terms are considered, already introduced for linear cases, indicating the importance of the cross-section area, in addition to moment of inertia in bending of thin beams. Those terms strongly increase the stiffness of the thin beam. The non-linear theory is applied to buckling problems of thin beams, especially in the study of the postbuckling behaviour.

1. Introduction

Higher order gradient and non-local elasticity theories proposed by Mindlin [1], Kroener [2] and Eringen [3], have been revisited by Aifantis [4] for solving boundary value problems, introducing the influence of the microstructure. Ru [5], proposed a simple approach to solve boundary value problems. Further applications to thin beams, thin films, micro-electromechanical systems and nano-electromechanical systems have been presented by Park and Gao [6] and Yang et al. [7].

Further applications in plasticity and dislocation dynamics may be found in [8,4], Fleck et al. [9–11]. Experimental evidence indicating the increase of the beam stiffness with the decrease of the thickness of the thin beam has been reported by Kakunai et al. [12] and Lam et al. [13]. Applications of nonlinear bending theory to thin plates have also been presented [14,15].

In the present work the nonlinear bending Bernoulli–Euler theory will be discussed into the context of a simplified strain gradient elasticity theory, where new terms, depending not only on the moment of inertia of the cross-section but also on the area of the cross-section are introduced. Those terms highly increase the stiffness of the beam especially when the beam is quite thin. Terms of the same type have been introduced by Yang et al. [7] and their theory has been applied to various bending problems [13,6,16]. Nevertheless, that couple stress theory, based upon an ad hoc assumption of zero double moments, does not include a substantial part of the strain gradient theory that is the increase of the higher order derivatives in the governing equilibrium equations. Those terms are necessary for the development of boundary layers which are characteristic of the strain gradient elasticity applications. It is pointed out that all the already presented strain gradient beam theories may be considered as ad hoc theories, since there are missing terms that their absence from the strain energy density may not be justified.

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0307-904X/$ - see front matter
http://dx.doi.org/10.1016/j.apm.2013.06.021

Lazopoulos and Lazopoulos [17] has proposed a bending theory including all the strain gradient terms of the Euler–Bernoulli strain. In fact those terms yield equilibrium equations depending not only upon the moment of inertia of the beam cross-section, as in the conventional beam theory, but also upon the area of the cross-section, that is quite important especially for thin beams. In the present work the strain gradient beam theory [17] has been extended just to include non-linear geometrical deformations. Indeed, non-linear curvature is employed just to introduce the geometrical non-linearities in the beam bending problem. With the help of straight forward perturbation expansions [18], the higher order bending problem has been described. The present non-linear expansion has applied to the non-linear beam buckling problem where the post-critical behaviour was studied. Application for the buckling problem is presented demonstrating the difference of the present theory from the conventional case.

2. Bending model of a strain gradient nonlinear elastic beam

A geometrically nonlinear version of Mindlin's elastic theory with microstructure is proposed. Following the practice from the conventional elasticity where Saint Venant's strain energy densities are employed for various geometrically non-linear problems, such as buckling problems [19], a widely used micro-elasticity theory equipped with two additional constitutive coefficients, apart from the Lame constants is adopted. The additional parameters are the bulk length g and the surface length l_k.

Indeed the strain energy density function, for the present case, is expressed by,

$$W = \frac{1}{2}\lambda\varepsilon_{mm}\varepsilon_{nn} + G\varepsilon_{mn}\varepsilon_{nm} + g^2\left(\frac{1}{2}\lambda\varepsilon_{kmm}\varepsilon_{knn} + G\varepsilon_{kmn}\varepsilon_{knm}\right) + l_k\left(\frac{1}{2}\lambda(\varepsilon_{kmm}\varepsilon_{nn} + \varepsilon_{mm}\varepsilon_{knn}) + G(\varepsilon_{kmn}\varepsilon_{nm} + \varepsilon_{mn}\varepsilon_{knm})\right), \tag{1}$$

where ε_{ij} denotes the nonlinear Green strain tensor $E = \frac{1}{2}(F^T F - I)$ where F stands for the deformation gradient, and ε_{ijk} the infinitesimal strain gradient respectively, with

$$\varepsilon_{ijk} = \varepsilon_{ikj} = \partial_i\varepsilon_{kj}, \tag{2}$$

and $u_i = u_i(x_k)$, the infinitesimal displacement field.

The constitutive Kirchhoff's stresses are defined by the relations,

$$\tau_{ij} = \frac{\partial W}{\partial\varepsilon_{ij}} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} + l_k(\lambda\varepsilon_{knn}\delta_{ij} + 2G\varepsilon_{kij}), \tag{3}$$

and the Kirchhoff's double-stresses by,

$$\mu_{ijk} = \frac{\partial W}{\partial\varepsilon_{ijk}} = g^2(\lambda\varepsilon_{inn}\delta_{jk} + 2G\varepsilon_{ijk}) + l_i(\lambda\varepsilon_{nn}\delta_{jk} + 2G\varepsilon_{jk}). \tag{4}$$

The x-axis denotes the axis of the beam, whereas the y axis indicates the deflection axis, see Fig. 1.

The elastic line lies on the x–y plane. Considering Bernoulli–Euler principle, strain of the beam is defined by,

$$\varepsilon_{xx} = -yk, \tag{5}$$

with k denoting the curvature of the elastic line.

For the formulation of the present problem we need the Kirchhoff stress

$$\tau_{xx} = \frac{\partial W}{\partial\varepsilon_{xx}} = E\varepsilon_{xx} + l_x E\varepsilon_{xxx}, \tag{6}$$

where E is the elastic Young's modulus and the Kirchhoff double-stresses

$$\mu_{xxx} = g^2 E\varepsilon_{xxx} + l_x E\varepsilon_{xx}, \tag{7}$$

$$\mu_{yxx} = g^2 E\varepsilon_{yxx}, \tag{8}$$

with the double-strains,

$$\varepsilon_{xxx} = -y\frac{\partial k}{\partial x} \quad \text{and} \quad \varepsilon_{yxx} = -k. \tag{9}$$

Fig. 1. Beam in x, y, z-axis.