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A method for analysis of linear dynamic systems driven by stationary non-Gaussian noise with applications to turbulence-induced random vibration

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ABSTRACT

A method is developed for approximating the properties of the state of a linear dynamic system driven by a broad class of non-Gaussian noise, namely, by polynomials of filtered Gaussian processes. The method involves four steps. First, the mean and correlation functions of the state of the system are calculated from those of the input noise. Second, higher order moments of the state are calculated based on Itô's formula for continuous semimartingales. It is shown that equations governing these moments are closed, so that moment of any order of the state can be calculated exactly. Third, a conceptually simple technique, which resembles the Galerkin method for solving differential equations, is proposed for constructing approximations for the marginal distribution of the state from its moments. Fourth, translation models are calibrated to representations of the marginal distributions of the state as well as its second moment properties. The resulting models can then be utilized to estimate properties of the state, such as the mean rate at which the state exits a safe set. The implementation of the proposed method is demonstrated by numerous examples, including the turbulence-induced random vibration of a flexible plate.

1. Introduction

Classical linear random vibration theory provides equations for calculating the first two moments of the state $\mathbf{X}(t)$ of a linear system subjected to input or driving noise characterized by its first two moments. The theory provides no information beyond the second moment properties of $\mathbf{X}(t)$ unless the noise is Gaussian, in which case the state $\mathbf{X}(t)$ is a Gaussian process. There are no efficient methods for calculating properties of $\mathbf{X}(t)$, and functionals of this process, for the general case of non-Gaussian driving noise.

This study develops a practical and efficient method for constructing approximate representations for the state $\mathbf{X}(t)$ of a linear dynamic system driven by a class of non-Gaussian noise that can be used to calculate properties or functionals of $\mathbf{X}(t)$. Developments are based on linear random vibration [1, Chapter 5], Itô's formula for continuous semimartingales [2, Section 4.6] an elementary solution for the problem of moments [3], and translation models $\mathbf{X}_T(t)$ for $\mathbf{X}(t)$ [4, Section 3.1.1]. Herein, we consider the driving noise to be from the class of non-Gaussian processes defined by polynomials of filtered

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Gaussian processes [5,6]. The Wiener homogeneous chaos [7,8] is a subset of this class of non-Gaussian process, which has been used extensively in applications (see, for example, [9–12]).

In previous work by the first author [6], the objective was to calculate the mean upcrossing rate of level x for scalar process $X(t)$ driven by a polynomial of a Gaussian process. Hermite approximations were developed for the joint density of $(X, \mathrm{d}X/\mathrm{d}t)$, and the approximations were used to find the mean upcrossing rate of level x. In the current study, the objective is to estimate the probability law of $\mathbf{X}(t)$, and the development is not limited to scalar-valued processes. It is assumed herein that $\mathbf{X}(t)$ can be approximated by a translation process, so that the marginal distribution of $\mathbf{X}(t)$ and its correlation function are needed. The construction of the marginal distribution is as in [6], generalized for the case of vector-valued processes. The construction of the correlation function for the input involves novel aspects of linear random vibration. Translation models are very flexible and have been used in a wide variety of modeling applications, including the dynamic response of a micro-electrical–mechanical system (MEMS) switch to random excitation [13], the seismic analysis of civil engineering structures [14], the material properties of foams [15] and for representing aggregates in concrete [16]. Additional applications include wind pressure fluctuations on bluff bodies [17], the response of geometrically nonlinear structures [18], the description of irregular masonry walls [19], and damage in glass plates [20].

Numerous examples are used to illustrate the proposed methodology. One of the examples demonstrates that the method can accurately predict the mean upcrossing rate for the state of a linear system driven by the square of an Ornstein–Uhlenbeck process. To demonstrate the method for a complex engineering application, the vibration response of a flexible plate subjected to turbulent flow is also presented. For this example, the random pressure fluctuations applied to the plate surface are proportional to the square of the velocity field, which is assumed to be Gaussian. Hence the applied pressure field is non-Gaussian.

The outline of the paper is as follows. In Section 2, we calculate correlation functions for both the input to linear systems and the state of these systems. Itô's formula is used subsequently to find higher order moments for the state of linear systems subjected to polynomials of filtered Gaussian processes. Translation models are constructed for the state of these systems in Section 3, and the use of the method to approximate the random vibration response of a flat plate subjected to turbulent flow is discussed in Section 4.

2. Correlation function and moments

Let $\mathbf{X}(t)$ be an $\mathbb{R}^d$-valued stochastic process defined by the following linear differential equation

$$\dot{\mathbf{X}}(t) = \mathbf{a}(t)\mathbf{X}(t) + \mathbf{b}(t)\mathbf{Z}(t), \quad t \geqslant 0, \tag{1}$$

where $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are $d \times d$ and $d \times d'$ matrices with real-valued, time-dependent entries, $\mathbf{Z}(t)$ denotes an $\mathbb{R}^{d'}$-valued input process, and $\mathbf{X}(0)$ is the initial state specified by its mean vector $\boldsymbol{\mu}_0 = \mathrm{E}[\mathbf{X}(0)]$ and covariance matrix, $\gamma_0 = \mathrm{E}[(\mathbf{X}(0) - \boldsymbol{\mu}_0)(\mathbf{X}(0) - \boldsymbol{\mu}_0)']$. Vector $\dot{\mathbf{X}}(t)$ has coordinates $\dot{X}_k(t) = \mathrm{d}X_k(t)/\mathrm{d}t$, $k = 1, \ldots, d$. It is assumed that input $\mathbf{Z}(t)$ is a weakly stationary process with mean $\boldsymbol{\mu}_{\mathbf{Z}} = \mathrm{E}[\mathbf{Z}(t)]$ and covariance function $\mathbf{c}_{\mathbf{Z}}(\tau) = \mathrm{E}[(\mathbf{Z}(t+\tau) - \boldsymbol{\mu}_{\mathbf{Z}})(\mathbf{Z}(t+\tau) - \boldsymbol{\mu}_{\mathbf{Z}})']$.

In Section 2.1, we present results from linear random vibration that describe the second-moment properties of $\mathbf{X}(t)$ defined by Eq. (1). Assuming the driving noise $\mathbf{Z}(t)$ is defined as a polynomial of a filtered Gaussian process, the second-moment properties of $\mathbf{Z}(t)$ are derived in Section 2.2. The equations for higher order moments of $\mathbf{X}(t)$ are developed in Section 2.3 for this class of input.

2.1. State second moment properties

Our objective in this section is to derive equations describing the time evolution of $\boldsymbol{\mu}(t) = \mathrm{E}[\mathbf{X}(t)]$ and $\mathbf{c}(t,s) = \mathrm{Cov}[\mathbf{X}(t), \mathbf{X}(s)]$, the mean and covariance of state $\mathbf{X}(t)$ described by Eq. (1). First, the expectation of Eq. (1) gives

$$\dot{\boldsymbol{\mu}}(t) = \mathbf{a}(t)\,\boldsymbol{\mu}(t) + \mathbf{b}(t)\,\boldsymbol{\mu}_{\mathbf{Z}}, \quad t \geqslant 0, \tag{2}$$

with initial condition $\boldsymbol{\mu}(0) = \boldsymbol{\mu}_0$. The difference between Eqs. (1) and (2) shows that the centered process $\tilde{\mathbf{X}}(t) = \mathbf{X}(t) - \boldsymbol{\mu}(t)$ satisfies the equation

$$\dot{\tilde{\mathbf{X}}}(t) = \mathbf{a}(t)\tilde{\mathbf{X}}(t) + \mathbf{b}(t)\tilde{\mathbf{Z}}(t), \quad t \geqslant 0, \tag{3}$$

where $\tilde{\mathbf{Z}}(t) = \mathbf{Z}(t) - \boldsymbol{\mu}_{\mathbf{Z}}$, $\mathrm{E}[\tilde{\mathbf{X}}(0)] = \mathbf{0}$, and $\mathrm{E}[\tilde{\mathbf{X}}(0)\tilde{\mathbf{X}}(0)'] = \gamma_0$. The centered process has zero mean and $\mathrm{E}[\tilde{\mathbf{X}}(t)\tilde{\mathbf{X}}(s)'] = \mathrm{Cov}[\mathbf{X}(t), \mathbf{X}(s)] = \mathbf{c}(t,s)$.

The solution to Eq. (3) is [21, Section 4.2]

$$\tilde{\mathbf{X}}(t) = \boldsymbol{\theta}(t,0)\,\tilde{\mathbf{X}}(0) + \int_0^t \boldsymbol{\theta}(t,s)\,\mathbf{b}(s)\,\tilde{\mathbf{Z}}(s)\,\mathrm{d}s, \quad t \geqslant 0, \tag{4}$$

where the $d \times d$ matrix $\boldsymbol{\theta}(t,s)$ is a system property satisfying the differential equation $\frac{\partial}{\partial t}\boldsymbol{\theta}(t,s) = \mathbf{a}(t)\,\boldsymbol{\theta}(t,s)$, $t \geqslant s$, with $\boldsymbol{\theta}(s,s)$ equal to the identity matrix $\forall s \geqslant 0$. For $t > s$, we have