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On the homotopy analysis method for solving a particle transport equation

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ABSTRACT

The purpose of this paper consists in the finding of the solution for a stationary neutron transport equation that is accompanied by the homogeneous boundary conditions, using the techniques of homotopy analysis method (HAM) and a numerical integration formula. Also, algorithm presented can be used for solving the integral–differential equations in which the unknown function depends on two variables, such as a radiative transfer equation. Results of a numerical example illustrate the accuracy and computational efficiency of the new proposed method.

1. Introduction

This paper deals with a typical problem of the mathematical-physics: the solving of a particle transport equation that has numerous applications in physics and astrophysics. In a reactor, the neutrons yielded at the fission of the nucleus are named the rapid neutrons and have a speed equals to 2×10^7 m/s. The rapid neutrons are subjected to a slowness process, their energy decreasing until these are in an equilibrium state with the others atoms of the environment. When the reactor is in a stationary state, the particles have the tendency to move from a region with a great density to another with a small density and thus on obtain a uniform density. The main problem in the nuclear reactor theory is to find the neutrons distribution in the reactor, hence its density, which is the solution of an integral–differential equation named the neutron transport equation. The resolution of the problems dealing with transport phenomena is the subject of several works: [1–7]. The following methods have been proposed in these papers: Fourier transform, Laplace transform, the least squares, the finite element method, Monte Carlo method, truncated series of Chebyshev polynomials, the fictitious domain method. A special attention has been given to the task of searching methods based on analytical procedures that generate accurate results to the transport problems. We mention: the spectral method, [8,2], the method proposed by Case and Hazeltine for S_N equations, [9,21], the spherical harmonics method, that expends the flux in Legendre polynomials, [10,11] and the techniques that solve the inverse radiative-transfer problems of homogeneous equations with the binomial scattering law and Henyey–Greenstein scattering law, [7]. The existence of solution to the problem of transport of particles was proved using functional analysis methods, [12,13].

The basic ideas of the homotopy, which is a concept of the topology and differential geometry, were used to obtain the approximate solutions for a wide class of differential, integral and integro-differential equations. The recent works [14,22] and [15], show that the solutions obtained by the homotopy perturbation method, variational iteration method and Adomian decomposition method are only special cases of the homotopy analysis method solutions. In this study, using the homotopy analysis method (HAM) established in 1992 by Liao [14] and the results obtained by the authors of the papers: [16–20,6,15], a new approach is presented to have the series solutions for an integro-differential equation. Unlike the works already

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published, the unknown function in our problem of particle transport depends on two variables. The comparative study between the solution obtained with the help of our algorithm for a numerical example with the appropriate exact solution leads to the conclusions regarding the values of the errors.

2. Problem formulation

Let us consider the integral–differential equation of transport theory for the stationary case:

$$\mu\frac{\partial\varphi(x,\mu)}{\partial x}+\sigma\varphi(x,\mu)=\frac{\sigma_s}{2}\int_{-1}^{1}\varphi(x,\mu')d\mu'+f(x,\mu)$$
$$\forall(x,\mu)\in D_1\times D_2=[0,1]\times[-1,1],\quad D_2=D_2'\cup D_2''=[-1,0]\cup[0,1], \tag{2.1}$$

with the following boundary conditions:

$$\varphi(0,\mu)=0\ \text{if}\ \mu>0,\quad \varphi(1,\mu)=0\ \text{if}\ \mu<0, \tag{2.2}$$

where

- $\varphi(x,\mu)$ is the density of neutrons, which migrate inside an isotropic medium toward a direction that makes the angle α with x axis and $\mu = \cos\alpha$;
- σ_s is the scattering coefficient, σ_a is the absorption coefficient, $\sigma=\sigma_s+\sigma_a$, for which there is a constant σ_0 with the property

$$\sigma-\sigma_s=\sigma_a\geqslant\sigma_0>0 \tag{2.3}$$

- $f(x,\mu)$ is a known particles source function.

Next we consider that the function φ belongs to a Hilbert space $L_2(D)$, where $D = D_1\times D_2$ and the norm is defined by the formula

$$\|\varphi\|=\left(\int_0^1\int_{-1}^{1}\varphi^2(x,\mu)dx\,d\mu\right)^{1/2}. \tag{2.4}$$

Theorem. *If $\varphi\in L_2(D)$ and $f\in L_2(D)$, then the solution of the problem (2.1) and (2.2) verifies the inequality*

$$\|\varphi\|\leqslant\frac{1}{\sigma_0}\|f\|. \tag{2.5}$$

Multiplying (2.1) by φ and integrating with respect to x and μ, we get

$$\int_0^1\int_{-1}^{1}\mu\frac{\partial\varphi(x,\mu)}{\partial x}\varphi(x,\mu)d\mu dx+\sigma\int_0^1\int_{-1}^{1}\varphi^2(x,\mu)d\mu dx$$
$$=\frac{\sigma_s}{2}\int_0^1\int_{-1}^{1}\left(\varphi(x,\mu)\int_{-1}^{1}\varphi(x,\mu')d\mu'\right)d\mu dx+\int_0^1\int_{-1}^{1}f(x,\mu)\varphi(x,\mu)d\mu dx \tag{2.6}$$

Using the integration by parts and the boundary conditions in the first integral, we have:

$$\int_0^1\int_{-1}^{1}\mu\frac{\partial\varphi(x,\mu)}{\partial x}\varphi(x,\mu)d\mu dx=\int_{-1}^{1}[\varphi^2(1,\mu)-\varphi^2(0,\mu)]\mu d\mu-\int_0^1\int_{-1}^{1}\mu\frac{\partial\varphi(x,\mu)}{\partial x}\varphi(x,\mu)d\mu dx$$
$$\Rightarrow 2\int_0^1\int_{-1}^{1}\mu\frac{\partial\varphi(x,\mu)}{\partial x}\varphi(x,\mu)d\mu dx=\int_{-1}^{0}\mu[\varphi^2(1,\mu)-\varphi^2(0,\mu)]d\mu$$
$$+\int_0^1\mu[\varphi^2(1,\mu)-\varphi^2(0,\mu)]d\mu=-\int_{-1}^{0}\mu\varphi^2(0,\mu)d\mu+\int_0^1\mu\varphi^2(1,\mu)d\mu\geqslant 0. \tag{2.7}$$

On the other hand, the Cauchy–Schwarz inequality leads to

$$\frac{\sigma_s}{2}\int_0^1\int_{-1}^{1}(\varphi(x,\mu)\int_{-1}^{1}\varphi(x,\mu')d\mu')d\mu dx=\frac{\sigma_s}{2}\int_0^1(\int_{-1}^{1}1\cdot\varphi(x,\mu)d\mu)^2dx\leqslant\frac{\sigma_s}{2}\int_0^1(\int_{-1}^{1}d\mu\int_{-1}^{1}\varphi^2(x,\mu)d\mu)dx$$
$$=\sigma_s\int_0^1\int_{-1}^{1}\varphi^2(x,\mu)d\mu dx. \tag{2.8}$$