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The *a posteriori* Fourier method for solving the Cauchy problem for the Laplace equation with nonhomogeneous Neumann data ☆

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ABSTRACT

In the present paper, the Cauchy problem for the Laplace equation with nonhomogeneous Neumann data in an infinite "strip" domain is considered. This problem is severely ill-posed, i.e., the solution does not depend continuously on the data. A conditional stability result is given. A new *a posteriori* Fourier method for solving this problem is proposed. The corresponding error estimate between the exact solution and its regularization approximate solution is also proved. Numerical examples show the effectiveness of the method and the comparison of numerical effect between the *a posteriori* and the *a priori* Fourier method are also taken into account.

1. Introduction

Consider the following Cauchy problem for the Laplace equation defined on the "strip" domain in $\mathbb{R}^{n+1}$:

$$\begin{cases} \Delta u(x,y) = 0, & x \in (0,1), \quad y \in \mathbb{R}^n, \quad n \geqslant 1, \\ u(0,y) = \varphi(y), & y \in \mathbb{R}^n, \\ u_x(0,y) = \psi(y), & y \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2}$ is a $(n+1)$-dimensional Laplacian operator.

This problem arises in many practical applications such as geophysics [1,2], non-destructive evaluation technique [3–6], cardiology [7], and etc. In the Hadamard's famous paper [8], this problem is firstly introduced as a classic example of ill-posed problems, which shows that any small change of the data may cause dramatically large errors in the solution. It is this ill-posedness that engenders peculiar problems in the numerical approximation and interpretation of solutions. If we only consider the solutions corresponding to the exact data, then the continuous dependence on the data can be recovered when some additional *a priori* bound on the unknown solution is imposed, this just is the so-called conditional stability. For this topic there have been many results, e.g. [9,4] and the references therein. However, the conditional stability cannot ensure the

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stability of numerical computation with noisy data which is usually obtained by measurement. Therefore, some effective regularization methods for solving this problem are necessary to be used to construct a stable solution. There also have been many papers devoted to this subject, e.g. the quasi-reversibility method [10–13], the Tikhonov regularization method [1,14], the variational method [15], the moment method [16–18], the wavelet method [19,20], the Fourier method [21], and etc. It's worth noting that most of the above papers consider *a priori* choice rule of the regularization parameter, which usually depends on both the *a priori* bound and the noise level. In general, the *a priori* bound cannot be known exactly in practice, and working with a wrong *a priori* bound may lead to a bad regularization solution.

Due to the linearity, the problem (1.1) can be divided into two problems with only one nonhomogeneous Dirichlet or Neumann data, respectively:

$$\begin{cases} \Delta v(x,y) = 0, & x \in (0,1), \quad y \in \mathbb{R}^n, \quad n \geqslant 1, \\ v(0,y) = \varphi(y), & y \in \mathbb{R}^n, \\ v_x(0,y) = 0, & y \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \Delta w(x,y) = 0, & x \in (0,1), \quad y \in \mathbb{R}^n, \quad n \geqslant 1, \\ w(0,y) = 0, & y \in \mathbb{R}^n, \\ w_x(0,y) = \psi(y), & y \in \mathbb{R}^n. \end{cases} \tag{1.3}$$

It is obvious that $u = v + w$ just is the solution of problem (1.1).

For the ill-posed problem (1.2), there have been many results, such as [21,20,19,22] and the reference therein.

Problem (1.3) is also severely ill-posed [2,23]. However, to our knowledge, there are few papers devoted to the error estimate of regularization methods except for [23] in which the methods of generalized Tikhonov regularization and generalized singular value decomposition are proposed. In addition, as the special case of Helmholtz equation, [24] has considered the *a priori* Fourier method for solving problem (1.3).

In this paper we only consider a new *a posteriori* Fourier method for solving problem (1.3). The advantage of the *a posteriori* method is that we do not need to know the smoothness and the *a priori* bound of unknown solution, in fact, they cannot be known exactly in practice. Note that the analysis of convergence rate for any *a posteriori* method can be given only under some assumptions on the exact solution, however, it is too hard to obtain convergence analysis for problem (1.3) by using standard information, such as the *a priori* bound given by the norm in $L^2(\mathbb{R}^n)$ or $H^p(\mathbb{R}^n)$. In order to overcome this difficulty, a novel *a priori* bound is introduced in Section 2, a conditional stability result is also given, meanwhile, the *a posteriori* choice rule of the regularization parameter and the corresponding error estimate between the exact solution and its regularization approximation are also provided in this section. In Section 3, some numerical examples are given, which show the effectiveness of the method. The comparison of numerical effect between the *a posteriori* and the *a priori* Fourier methods is also taken into account in this section. The paper ends with a brief conclusion in Section 4.

2. The conditional stability and the *a posteriori* Fourier regularization for solving problem (1.3)

It is easy to know as in [24], the solution of problem (1.3) is given by

$$\hat{w}(x,\xi) = \frac{\sinh(x|\xi|)}{|\xi|}\hat{\psi}(\xi), \tag{2.1}$$

or equivalently

$$w(x,y) = \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{iy\cdot\xi}\frac{\sinh(x|\xi|)}{|\xi|}\hat{\psi}(\xi)d\xi, \tag{2.2}$$

where $\hat{w}(x,\xi)$ denotes the Fourier transform of function $w(x,y)$ with respect to variable $y \in \mathbb{R}^n$.

Noting that the factor $\frac{\sinh(x|\xi|)}{|\xi|}$ in (2.1) and (2.2) increases exponentially as $|\xi| \to +\infty$, so the problem is severely ill-posed. In fact, in practice the Neumann data $\psi(y)$ is given only by measurement. Assume that the exact data $\psi(y)$ and the noisy data $\psi_\delta(y)$ both belong to $L^2(\mathbb{R})$ and satisfy the following noise level

$$\|\psi(\cdot) - \psi_\delta(\cdot)\| \leqslant \delta, \tag{2.3}$$

where $\|\cdot\|$ denotes the norm in $L^2(\mathbb{R}^n)$. This error must be amplified infinitely by the factor $\frac{\sinh(x|\xi|)}{|\xi|}$ and which leads to blow up of integral (2.2) with the noisy data $\psi_\delta(y)$.

The following *a priori* bound is used in [24]:

$$\|w(1,\cdot)\|_p \leqslant E_2, \quad p \geqslant 0, \tag{2.4}$$