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On zero duality gap in surrogate constraint optimization: The case of rational-valued functions of constraints

Bahram Alidaee [a,*], Haibo Wang [b,1]

[a] School of Business Administration, The University of Mississippi, University, MS 38677, USA
[b] Division of International Business and Technology Studies, A.R. Sanchez School of Business, Texas A&M International University, Laredo, TX 78041, USA

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ABSTRACT

This paper is concerned with the constrained optimization problem. A detailed discussion of surrogate constraints with zero duality gaps is presented. Readily available surrogate multipliers are considered that close the duality gaps where constraints are rational-valued. Through illustrative examples, the sources of duality gaps are examined in detail. While in the published literature, in many situations conclusions have been made about the existence of non-zero duality gaps, we show that taking advantage of full problem information can close the duality gaps. Overlooking such information can produce shortcomings in the research in which a non-zero duality gap is observed. We propose theorems to address the shortcomings and report results regarding implementation issues.

1. Introduction

Consider the optimization problem (P) given as:

$$P : Min_{x \in X} f(x) : \text{ st. } g(x) \leqslant 0, \tag{1}$$

where f and each component $g_i(x)$ of the vector $g(x)$ are real-valued functions defined on X. No specific characteristics of these functions or of X are assumed unless otherwise specified. Let $X(F)$ to be the set of all feasible points in P defined as:

$$X(F) = \{x \in X : g(x) \leqslant 0\}.$$

No distinction is made between the row and column vectors, and all vector products are dot products in the usual sense and conformable dimensions are taken for granted. As an alternative to Lagrangian relaxation, Glover [1] introduced the *surrogate constraint* relaxation. A surrogate constraint for problem P is a linear combination of the component constraints of $g(x) \leqslant 0$ that associates a multiplier u_i with each $g_i(x) \leqslant 0$ to produce the single inequality $ug(x) \leqslant 0$, where $u = (u_i)$. This inequality is implied by $g(x) \leqslant 0$ whenever $u \geqslant 0$ [2]. Given a multiplier vector $u \geqslant 0$ the surrogate problem is then defined by:

$$SP(u) : Min_{x \in X} f(x) : \text{ st. } ug(x) \leqslant 0. \tag{2}$$

Denote the optimal objective function value for $SP(u)$ to be $s(u)$ defined by:

$$s(u) : inf_{x \in X(u)} f(x) : \quad \text{where } X(u) = \{x \in X : ug(x) \leqslant 0\}. \tag{3}$$

* Corresponding author. Tel.: +1 662 915 5466.
E-mail addresses: Balidaee@bus.olemiss.edu (B. Alidaee), hwang@tamiu.edu (H. Wang).
[1] Tel.: +1 956 326 2503.

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Because $SP(u)$ is a relaxation of P (for $u \geqslant 0$), $s(u)$ cannot exceed the optimum objective function value for P and approaches this value more closely as $ug(x) \leqslant 0$ becomes a more '*faithful*' representation of the constraint $g(x) \leqslant 0$. Obviously we always have $X(F) \subseteq X(u)$ and thus a faithful representation of $g(x) \leqslant 0$ by $ug(x) \leqslant 0$ depends upon how large the set $X(u)$ is compared to the set $X(F)$. Choices of the vector u that improve the proximity of $SP(u)$ to P, i.e., that provide the greatest values of $SP(u)$, yield the *strongest* surrogate constraints which motivated the definition of the surrogate dual SD as follows:

$$SD : Max_{u \geqslant 0} s(u). \tag{4}$$

Since $X(F) \subseteq X(u)$ then optimal value of surrogate dual SD is greater than or equal to optimal value of problem P. The amount of difference is called surrogate duality gap. The smaller this gap is the more faithfully the single inequality $ug(x) \leqslant 0$ represents the system of inequalities $g(x) \leqslant 0$.

An important observation in surrogate dual problem SD is that, except for non-negativity there is no restriction on the values of the multiplier vectors u. Components u_i of u can be any real number, i.e., integer, rational or irrational.

An early theoretical work on surrogate optimization by [3], reported several major results. One of their contributions was to observe that the optimal value of the SD problem is always as good as the optimal value of the well-known Lagrangian dual LD defined by:

$$LD : Max_{u \geqslant 0} L(u), \tag{5}$$

where $L(u)$ is the function given by:

$$L(u) : inf_{x \in X}\{f(x) + ug(x)\}. \tag{6}$$

This result thus certifies that the surrogate constraint duality gap is always as small or smaller than the Lagrangian duality gap.

Over the years researchers have introduced a variety of different methods for finding the multiplier vectors u that yield the strongest surrogate constraints (see, for example, Refs. [4–10], among others.) However, except for a few papers [11–18], research has been restricted to rational-valued and in many cases integer valued multipliers. We will show such restriction is a major source for existence of the surrogate duality gap in a wide variety of problems which is not discussed in the previous research literature.

Obviously an important issue in surrogate constraint optimization is the way that constraints are aggregated to create a single constraint. If it is beneficial to the search process, several constraints may be created instead of a single constraint; however, without loss of generality, in the paper we concentrate on the creation of a single constraint. We take our motivation and focus by marrying the treatment of surrogate constraints formed from problem inequalities with approaches based on constraint aggregation, which deals with equations rather than inequalities. There is some ambiguity in the literature in referring to these different kinds of approaches, which we clarify subsequently. Most constraint aggregation schemes for IP group two constraints (equations) into one and then sequentially aggregate each of the remaining constraints with the newly formed constraint [19]. Methods for simultaneously combining constraints (both equations and inequalities) also have been introduced by several researchers [20,5,19]. In all of these methods the concern is to find a set of multipliers to aggregate multiple constraints into a single one. Evidently, if a solution to the single constraint problem also satisfies all constraints of the problem P then we have an optimal solution to the original problem, i.e., we have a zero duality gap. Based on the aforementioned discussion, an important research topic is to find a multiplier (or a set of multipliers) u that the set of solutions to the single constraint $ug(x) \leqslant 0$ is the same as the set of solutions to the system of constraints $g(x) \leqslant 0$, i.e., $X(F) = X(u)$. We should emphasize that in order to have a zero surrogate duality gap it is unnecessary to have $X(F) = X(u)$, however it is a sufficient condition.

To the best of our knowledge there are at least five research streams that attempt to close the duality gap in surrogate constraint methods. In the context of 'equalities', finding an aggregation of a given set of equalities to create a single one (or several equalities) with the same set of solutions as the original system of equations possess has been an important research topic at least for the last 110 years. The seminal paper by Matthews [21] probably is the first publication to provide a solution to this problem. He showed a system of two equations expressed in the form:

$$\sum_{j=1}^{n} a_{1j}x_j = b_1, \quad \sum_{j=1}^{n} a_{2j}x_j = b_2, \tag{7}$$

where the $x_j \geqslant 0, j = 1,\ldots,n$, are unknown integers, and a_{ij} and b_i are given positive integers possess the same set of solutions as the single equation:

$$\sum_{j=1}^{n}(u_1a_{1j} + u_2a_{2j})x_j = u_1b_1 + u_2b_2, \tag{8}$$

provided that u_1 and u_2 are suitably chosen *relatively prime* integers (whose greatest common divisor is one). Furthermore, Mathews extended the result to more than two equations. During the last four decades researchers have put major emphasis to come up with a set of multipliers u to create a single equation by combining several equations where the set of solutions to the single equation is the same as the set of solutions to the original set of equations, see survey paper [19]. However, all previous results impose many limiting assumptions such as: variables and equations must be bounded, variables must be