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A novel mathematical modeling of multiple scales for a class of two dimensional singular perturbed problems

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ABSTRACT

A novel mathematical modeling of multiple scales (NMMMS) is presented for a class of singular perturbed problems with both boundary or transition layers in two dimensions. The original problems are converted into a series of problems with different scales, and under these different scales, each of the problem is regular. The rational spectral collocation method (RSCM) is applied to deal with the problems without singularities. NMMMS can still work successfully even when the parameter ε is extremely small ($\varepsilon = 10^{-25}$ or even smaller). A brief error estimate for the model problem is given in Section 2. Numerical examples are implemented to show the method is of high accuracy and efficiency.

1. Introduction

Singular perturbed problems (SPPs) are of common occurrence in many branches of applied mathematics, such as fluid dynamics, elasticity, chemical reactor theory, etc. Some previous work for SPPs includes the method of matched asymptotic expansion, the method of multiple scales, the method of the strained coordinate, the method of averaging, and the method of boundary layer correction, etc. (more details see [1–7]).

Parameterized singular perturbed two-point boundary value problems with a boundary layer are consider in [8,9]. By the boundary layer correction technique, the original problem is converted into two non-singularly perturbed problems which can be solved using traditional numerical methods. One is the reduced problem, and the other is named the modified problem. Numerical examples show that the method is of high accuracy and efficiency when dealing such problems. However, the solution of the reduced problem must be regular in the whole computational domain, and be accurate in the areas away from the boundary layers in [8,9]. Therefore, this method meets great difficulties when dealing with problems with transition layers. For example, the solution of the reduced problem may be have singularities, such as existing transition layers. Thus, the boundary layer correction method has some difficulties when dealing with problems with transition layers.

The method of matched asymptotics [1,10–13] is useful for deriving approximate solutions of partial differential equations. To apply the method, the domain is divided into two or more overlapping regions in which different asymptotic limits are valid. The asymptotic solutions valid in the regions are matched in the regions of overlap to obtain a global solution. For many applications, the domain consists of the inner and outer regions with the inner region often being near a boundary or singularity and usually being much smaller than the outer region.

Numerical analysis and asymptotic analysis are two principle approaches for solving singular perturbation problems. Since the goals and the problem classes are rather different, these has not been much interaction between these approaches.

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In the present paper, we propose a novel method which bridge these approaches. This approach is based on the boundary layer correction technique and interior layer correction technique. By constructing some modified problems, the original problem can be converted into a series of non-singularly perturbed problems with different scales, which can easily be solved by using classic numerical methods, such as the rational spectral collocation method. Thus, we can solve singular perturbed problems effectively no matter how small the perturbed parameter ε is.

The differential quadrature method (DQM) was first introduced by Bellman and Casti [14,15]. The DQM is basically equivalent to the spectral collocation method, and if the computation domain is regular, DQM can deal with boundary conditions easily to yield high accuracy with little computation. Compared with DQM, the rational spectral collocation method has smaller round-off error, especially when N is very large [16–19].

For a type of linear problems, an error estimate is given in Section 2. The analysis for general problems with transition layers are really complicated. We give some numerical examples to show the processes of our method. From the numerical examples, we can see that the method proposed in this paper is of high efficiency and accuracy. For the detailed discussions for general problems with transition layers, the interested readers are referred to [20–22].

The remainder of this paper is organized as follows. In Section 2, the case with two boundary layers is considered. In Section 3, the case with two boundary layers and one transition layer is discussed. In Section 4, the case with Burgers' equation with oblique transition layers is introduced to show that NMMMS can be applied to a type of nonlinear problems successfully. Finally, in Section 5, a short conclusion is presented.

2. The case with two boundary layers

2.1. Model problem

In this section, we consider a class of singular perturbed problems as follows

$$\varepsilon\Delta u + a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), \quad (x,y) \in \Omega, \tag{2.1a}$$

$$u|_{\partial\Omega} = u_{\partial\Omega}, \tag{2.1b}$$

where ε is a small positive parameter, $\Omega = (0,1)^2$. The case when $\varepsilon \ll 1$ in (2.1) is particularly interesting and challenging. Many different phenomenas can arise in such problems, including boundary layers and complicated interval transition regions. In this section, suppose $a(x,y) \geqslant d > 0$, $b(x,y) \geqslant d > 0$, $c(x,y) \leqslant 0$, with d to be a positive constant. Furthermore, to simplify the description, suppose

1. u has only two boundary layers which are located at $y = 0$, $x = 0$, respectively;
2. the order of the boundary layers is $\mathcal{O}(\varepsilon)$;
3. u is bounded in Ω.

Under these assumptions, denote.

$$\xi = \frac{x}{\varepsilon}, \tag{2.1b}$$

$$\eta = \frac{y}{\varepsilon}. \tag{2.1c}$$

Denote $u^\varepsilon(x,y)$ to be an approximation of $u(x,y)$ in Ω, which can be expressed by

$$u^\varepsilon(x,y) = w(x,y) + v^1(\xi,y) + v^2(x,\eta) + v^3(\xi,\eta). \tag{2.1d}$$

Substitution of (2.1d) into the left side of (2.1), we have

$$L[u^\varepsilon] = \varepsilon\Delta u^\varepsilon + a(x,y)u^\varepsilon_x + b(x,y)u^\varepsilon_y + c(x,y)u^\varepsilon \tag{2.5a}$$
$$= \varepsilon\Delta w + a(x,y)w_x + b(x,y)w_y + c(x,y)w \tag{2.5b}$$
$$+ \frac{1}{\varepsilon}v^1_{\xi\xi} + \varepsilon v^1_{yy} + \frac{a(x,y)}{\varepsilon}v^1_\xi + b(x,y)v^1_y + c(x,y)v^1 \tag{2.5c}$$
$$+ \frac{1}{\varepsilon}v^2_{\eta\eta} + \varepsilon v^2_{xx} + \frac{b(x,y)}{\varepsilon}v^2_\eta + a(x,y)v^2_x + c(x,y)v^2 \tag{2.5d}$$
$$+ \frac{1}{\varepsilon}(v^3_{\xi\xi} + v^3_{\eta\eta}) + \frac{a(x,y)}{\varepsilon}v^3_\xi + \frac{b(x,y)}{\varepsilon}v^3_\eta + c(x,y)v^3. \tag{2.5e}$$

Let $w(x,y)$, $v^1(\xi,y)$, $v^2(x,\eta)$, $v^3(\xi,\eta)$ satisfy P_1–P_4, respectively.
P_1:

$$a(x,y)w_x + b(x,y)w_y + c(x,y)w = f(x,y), \tag{2.6a}$$