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The general Jacobi matrix method for solving some nonlinear ordinary differential equations

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A B S T R A C T

In this paper, we obtain the approximate solutions for some nonlinear ordinary differential equations by using the general Jacobi matrix method. Explicit formulae which express the Jacobi expansion coefficients for the powers of derivatives and moments of any differentiable function in terms of the original expansion coefficients of the function itself are given in the matrix form. Three test problems are discussed to illustrate the efficiency of the proposed method.

1. Introduction

Approximations by orthogonal polynomials have played a main role in the development of physical sciences, engineering and mathematical analysis. Techniques for finding approximate solutions for differential equations, based on the classical orthogonal polynomials, are popularly known as spectral methods [1–5]. Approximating functions in spectral methods are related to polynomial solutions of eigenvalue problems in ordinary differential equations, known as Sturm–Liouville problems. In the past few decades, there has been growing interest in this subject. As a matter of fact, the spectral methods provide a competitive alternative to other standard approximation techniques, for a large variety of problems. The first applications of this method were concerned with the investigation of periodic solutions of boundary value problems using trigonometric polynomials. Expansions in orthogonal basis functions were performed, due to their high accuracy and flexibility in computations, see, for instance, the intensive seminal work of Doha [6–9] and Doha and Ahmed [10]. It is worth to be noted that one of the important and interesting implementations of this technique is a matrix presentation. For example Sezer and Kaynak presented a Chebyshev matrix method to solve linear differential equations [11]. Also in [12] Sezer and Dogan used this method for obtaining approximate solutions of linear and non-linear Fredholm integral equations. Furthermore Köroğlu [13] proposed this method for solving Fredholm integro-differential equations. Also we refer the interested reader to [14–18] for some semi-analytical approaches for the numerical solution of nonlinear differential equations.

In this paper, using the general Jacobi matrix method, we obtain the approximate solutions of some nonlinear ordinary differential equations. In Section 2 we introduce the Jacobi polynomials and their properties. Section 3 is allocated to obtain

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the matrix relations for the powers of derivatives and moments of the expansion of Jacobi polynomials in terms of original Jacobi polynomials. In Section 4 we report our numerical finding to demonstrate the accuracy and efficiency of the proposed method by considering three test problems. Suppose $R_{i,j}(x)$ are functions that have Taylor series.

In this paper we are interested to find the Jacobi approximations for solutions of the class of some nonlinear ordinary differential equations in the form

$$\begin{aligned}&\sum_{k=1}^{s_j} R_{k,j}(y^{(j)})^k + \sum_{k=1}^{s_{j-1}} R_{k,j-1}(y^{(j-1)})^k + \cdots + \sum_{k=1}^{s_0} R_{k,0}(y^{(0)})^k = f(x),\\ &-1 \leqslant x \leqslant 1, \quad j \geqslant 0,\\ &s_p > 0, \quad p = 0,\ldots,j,\end{aligned} \tag{1}$$

subject to

$$\sum_{i=0}^{j} b_{i,l} y^{(l)}(a_i) = \theta_l, \quad a_i \in [-1,1], \quad l = 0,\ldots,j-1, \quad i = 0,\ldots,j. \tag{2}$$

Firstly, we write Taylor series of $R_{k,j}(x)$ in the form:

$$R_{k,j}(x) \simeq \sum_{i=0}^{m_j} r_{k,i}^{(j)} x^i, \tag{3}$$

where $m_p > 0$, $(p = 0,1,\ldots,j)$ are integer numbers.

Putting (3) in (1) yields:

$$\sum_{k=1}^{s_j}\sum_{i=0}^{m_j} r_{k,i}^{(j)} x^i (y^{(j)}(x))^k + \sum_{k=1}^{s_{j-1}}\sum_{i=0}^{m_{j-1}} r_{k,i}^{(j-1)} x^i (y^{(j-1)}(x))^k + \cdots + \sum_{k=1}^{s_0}\sum_{i=0}^{m_0} r_{k,i}^{(0)} x^i (y(x))^k \simeq f(x). \tag{4}$$

Now suppose we consider the Jacobi approximation of the exact solution of (1) in the form:

$$y(x) \simeq \sum_{i=0}^{n} \delta_i^{(0,0,1)} P_i^{(\alpha,\beta)}(x) = \left(\Delta_n^{(0,0,1)}\right)^T P^{(\alpha,\beta)}(x), \tag{5}$$

then $x^i(y^{(j)}(x))^k (i,j \geqslant 0, k > 0)$ could be written as:

$$x^i (y^{(j)}(x))^k \simeq \sum_{t=0}^{n} \delta_t^{(i,j,k)} P_t^{(\alpha,\beta)}(x) = \left(\Delta_{0:n}^{(i,j,k)}\right)^T P^{(\alpha,\beta)}(x), \tag{6}$$

where

$$\Delta_{p:q}^{(i,j,k)} = [\delta_p^{(i,j,k)}, \delta_{p+1}^{(i,j,k)}, \ldots, \delta_q^{(i,j,k)}]^T; \quad (q > p); \; \Delta_{0:q}^{(i,j,k)} = \Delta_q^{(i,j,k)}, \tag{7}$$

and

$$P^{(\alpha,\beta)}(x) = \left[P_0^{(\alpha,\beta)}(x), P_1^{(\alpha,\beta)}(x), P_2^{(\alpha,\beta)}(x), \ldots, P_n^{(\alpha,\beta)}(x)\right]^T. \tag{8}$$

The main goal of this paper is obtaining the coefficients $\delta_t^{(0,0,1)}; t = 0, 1 \ldots n$ (the coefficients of Jacobi approximation (5)). In the next sections we study the method of obtaining unknowns $\delta_t^{(0,0,1)}$. However before that we introduce some properties of Jacobi polynomials.

2. Some properties of Jacobi polynomials

The Jacobi polynomials associated with the real parameters $(\alpha > -1, \beta > -1)$, are a sequence of polynomials $P_n^{(\alpha,\beta)}(x)$, $(n = 0,1,2,\ldots)$, satisfying the orthogonality relation [19];

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) dx = \begin{cases} 0, & m \neq n, \\ h_n, & m = n, \end{cases} \tag{9}$$

where

$$h_n = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}. \tag{10}$$

These polynomials are eigenfunctions of the following singular Sturm–Liouville equation [19]:

$$(1-x^2)\Phi''(x) + [\beta - \alpha - (\beta+\alpha+2)x]\Phi'(x) + n(n+\beta+\alpha+1)\Phi(x) = 0. \tag{11}$$