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E-Bayesian estimation of the reliability derived from Binomial distribution ☆

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ABSTRACT

This paper introduces a new parameter estimation method, *named E-Bayesian estimation method*, to estimate reliability derived from Binomial distribution. The definition of E-Bayesian estimation of the reliability is proposed, the formulas of E-Bayesian estimation and hierarchical Bayesian estimation of the reliability are also provided. Finally, it is shown, through a numerical example, that the new method is much simpler than hierarchical Bayesian estimation in practice.

1. Introduction

Lindley and Smith [1] first introduced the idea of hierarchical prior distribution. Han [2] developed methods to construct hierarchical prior distribution. Recently, hierarchical Bayesian methods have been applied to data analysis [3]. However, in all of these methods the integration is inevitable, which is not easy in general, though some computing methods such as MCMC (Markov chain Monte Carlo) are available [4,5]. On the contrary, we will see in the following sections that the E-Bayesian estimation method is very simple.

In some situations, it is hard to determine the types of life distributions of the products. Sometimes, only failure numbers could be obtained, whereas the accurate failure times could not be obtained, though the life distributions are aware. In these situations, we can get the reliability estimate by means of the non-parameter method. For instance, suppose the life distribution of a certain kind of product is unaware, we choose products randomly to do the type I censored life test.

Suppose X is the number of failures in n independent trials, R is the probability of a success at each individual trial, then X is a random variable with Binomial distribution, thus

$$P(X=r)=\binom{n}{r}R^{n-r}(1-R)^{r},\quad 0<R<1,\quad r=0,1,\ldots,n, \tag{1}$$

where R is also called the reliability of product at the censored time.

Now, the research on non-parameter estimation of the reliability is changed to that on the parameter estimation in Binomial distribution. About the study for reliability with Binomial distribution may be found in [6–8].

The definition and the formulas of E-Bayesian estimation of the reliability are proposed in Section 2 and Section 3 respectively. In Section 4, formulas of hierarchical Bayesian estimation of the reliability are provided. In Section 5, a numerical example is given. Section 6 is the conclusions.

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2. Definition of E-Bayesian estimation of *R*

We take the conjugate prior of R as $Beta(a,b)$, with density function

$$\pi(R|a,b) = R^{a-1}(1-R)^{b-1}/B(a,b), \quad 0 < R < 1, \tag{2}$$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ is the beta function, and hyperparameters $a > 0$, $b > 0$.

The derivative of $\pi(R|a,b)$ with respect to R is

$$\frac{d\pi(R|a,b)}{dR} = R^{a-2}(1-R)^{b-2}[(a-1)(1-R)-(b-1)R]/B(a,b).$$

According to Han [2], a and b should be chosen to guarantee that $\pi(R|a,b)$ is a increase function of R. Thus $a > 1$, $0 < b < 1$.

Given $b = 1$, and the larger the value of a, the thinner the tail of the density function is. Berger [9] had shown that the thinner tailed prior distribution often reduces the robustness of the Bayesian estimate. Consequently, the hyperparameter a should be chosen under the restriction $1 < a < c$, where c is a given upper bound (c is a positive constant). How to determine the constant c would be described later in an example (the key point to consider is the robustness of E-Bayesian estimates).

In this paper we only consider the case when $b = 1$. In this case, the density function $\pi(R|a,b)$ becomes

$$\pi(R|a) = aR^{a-1}, \quad 0 < R < 1. \tag{3}$$

For parameter of exponential distribution, the definition of E-Bayesian estimation was originally addressed by Han [10]. For Binomial distribution, the E-Bayesian estimation of the reliability is defined as follows:

Definition 1. With $\widehat{R}_B(x)$ being continuous,

$$\widehat{R}_{EB} = \int_D \widehat{R}_B(a)\pi(a)\,da$$

is called the expected Bayesian estimation of R (briefly E-Bayesian estimation),which is assumed finite, where D is the domain of a, $\widehat{R}_B(a)$ is Bayesian estimation of R with hyperparameter a, and $\pi(a)$ is the density function of a over D.

Definition 1 indicates that the E-Bayesian estimation of R is just the expectation of the Bayesian estimation of R for the hyperparameter.

3. E-Bayesian estimation of *R*

In this section, three different prior distributions of the hyperparameter are taken to investigate the influence of different prior distributions on the E-Bayesian estimation of R.

Theorem 1. *For Binomial distribution (1), if the prior density function $\pi(R|a)$ of R is given by (3) , then*

(i) With the quadratic loss function, the Bayesian estimation of R is

$$\widehat{R}_B(a) = \frac{a+n-r}{a+n+1};$$

(ii) For the following priors of a

$$\pi_1(a) = \frac{2(c-a)}{(c-1)^2}, \quad 1 < a < c, \tag{4}$$

$$\pi_2(a) = \frac{1}{c-1}, \quad 1 < a < c, \tag{5}$$

$$\pi_3(a) = \frac{2a}{c^2-1}, \quad 1 < a < c \tag{6}$$

the corresponding E-Bayesian estimation of R are, respectively

$$\widehat{R}_{EB1} = 1 - \frac{2(r+1)}{(c-1)^2}\left\{(n+c+1)\ln\left(\frac{n+c+1}{n+2}\right) - (c-1)\right\},$$

$$\widehat{R}_{EB2} = 1 - \frac{(r+1)}{(c-1)}\ln\left(\frac{n+c+1}{n+2}\right),$$

$$\widehat{R}_{EB3} = 1 - \frac{2(r+1)}{c^2-1}\left\{(c-1) - (n+1)\ln\left(\frac{n+c+1}{n+2}\right)\right\}.$$