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Steady state visco-elastic rimming flow

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ABSTRACT

Using scale analysis and the method of perturbations, a theoretical description is obtained for the steady-state non-Newtonian flow on the inner wall of the rotating horizontal cylinder. The Maxwell upper-convective equation is chosen to model the visco-elastic properties of the fluid. In the general case, the derived governing equations can be solved only numerically. However, since the polymeric solutes used in roto-molding and coating technologies exhibit the relatively weak elastic properties, the Deborah number for such flows is rather small ($De < 1$). Exploiting this fact, the perturbation method is applied for simplification of the model. As a result, the first order non-linear differential equation for the thickness of the fluid film is derived. An approximate analytical solution of this equation is found. The accuracy of analytical solution is verified by the direct numerical solution of the derived equation. The obtained equation is rather complex and contains several critical points. These points are classified by the analysis of the corresponding autonomous system. The type and location of these critical points are accounted for during numerical solution of the equation. Using the obtained solutions, the criteria which guarantee the stable steady-state flow of the liquid polymer and the uniform final thickness of the coating film are determined. The bounds for the different flow regimes and principal controlling parameters are identified.

1. Introduction

The problem of rotational flow on the inner and/or outer wall of a hollow horizontal cylinder has been of interest for many years due to its wide range of applications in industry [1,2]. Moffatt [3] was the first to derive the condition of the maximal supportable load for a Newtonian liquid. Later Preziosi and Joseph [4] presented the same condition in another form and named it a run-off condition for coating and rimming flows. The possible instability of the liquid film on a cylindrical surface is one the most challenging fundamental aspects of this problem. A highly unstable nature of rimming Newtonian flow was discussed in a number of recent publications [5–10]. For example, Benilov and O'Brien [5] and Benilov [6] examined the stability of solutions, accounting for inertia and surface tension, and concluded that including these higher order corrections to the governing equation for the liquid film thickness may cause the instability of the steady-state solution. They proved that inertia always causes instability, but viscosity can make the characteristic time of growth large enough to effectively stabilize the film. Benilov et al. [7,8] have shown that the system admits strongly unstable solutions, which develop singularities in a finite time.

Although the aforementioned investigations highlight the main characteristics of the rimming flow, given its importance, not enough has been done to show the effect of non-Newtonian properties on such flow. Only a few attempts have been

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Nomenclature

De	Deborah number
C_B	inverse to the Bond number
$\mathbf{e}$	rate of deformation tensor
$\mathbf{e}_r$, $\mathbf{e}_\theta$	radial and azimuthal axes vectors, respectively
$\mathbf{g}$	gravity vector
H_0	mean thickness of the liquid layer
h	thickness of the liquid layer
h_0	characteristic thickness of the liquid layer
$\mathbf{n}$	normal to the free surface
p	pressure
q	mass flux
r	radial coordinate
r_o	radius of the cylinder
R	non-dimensional radial coordinate = $(1-r)/\delta$
Re	Reynolds number as defined by equation
t	time
$\mathbf{v}$	fluid velocity
v_r, v_θ	radial and azimuthal components of the fluid velocity, respectively
W	total mass of the liquid
Greek symbols	
δ	ratio of the characteristic liquid layer thickness and radius of the cylinder (defined by Eq. (5))
κ	mean curvature of the free surface
λ	parameter in stress–strain constitutive equation (relaxation time)
μ	parameter in stress–strain constitutive equation (dynamic viscosity)
θ	azimuthal coordinate
ρ	liquid density
σ	surface tension
$\boldsymbol{\tau}$	deviator of the stress tensor
$\tau_{\theta R}$, τ_{RR}, $\tau_{\theta\theta}$	components of $\boldsymbol{\tau}$
Ω	angular velocity of the cylinder
Superscripts	
$*$	dimensional quantities
0, 1	zero-order and first-order approximation, respectively
Subscripts	
0	characteristic quantity
θ, r	azimuthal and radial components, respectively

made, in which power-law model [11], Carrea–Yasuda model [12,13], Ellis model [14], and Bingham model [15] were used. The visco-elastic model was studied in [16] and the corresponding governing equations were solved numerically.

Most polymeric solutes used in rotational coating are non-Newtonian liquids, which exhibit moderate elastic behavior, which is characterized by the Deborah number $De = \lambda\Omega$, where Ω is the characteristic angular velocity of the rotating cylinder and λ is a typical time of relaxation for liquid polymers. The values of the Deborah number are well documented [16–18] and normally stay in the range from 10^{-2} to 10. Our main concern is the rotational molding of highly viscous polymers [1,2] that exhibit non-Newtonian visco-elastic behavior. We are particularly interested in eliminating possible instabilities and providing the criteria for the steady-state flow in order to obtain a continuous and smooth coating film on the wall of the horizontal cylinder. In the steady-state case, the equation that describes the distribution of the liquid film along the wall of the rotating cylinder is a non-linear ordinary differential equation, which in general may have an infinite number of solutions. In the present study of the rimming flow of the viscoelastic non-Newtonian fluid the governing equation is solved numerically and analytically.

2. System model and scale analysis

A schematic sketch of the rimming flow is illustrated in Fig. 1. The cylinder of radius r_0 is rotating in the counterclockwise direction with constant rate Ω. The horizontal cylinder is assumed to be of infinite length and is open. A highly viscous liquid