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Wavelets collocation methods for the numerical solution of elliptic BV problems

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ABSTRACT

Based on collocation with Haar and Legendre wavelets, two efficient and new numerical methods are being proposed for the numerical solution of elliptic partial differential equations having oscillatory and non-oscillatory behavior. The present methods are developed in two stages. In the initial stage, they are developed for Haar wavelets. In order to obtain higher accuracy, Haar wavelets are replaced by Legendre wavelets at the second stage. A comparative analysis of the performance of Haar wavelets collocation method and Legendre wavelets collocation method is carried out. In addition to this, comparative studies of performance of Legendre wavelets collocation method and quadratic spline collocation method, and meshless methods and Sinc–Galerkin method are also done. The analysis indicates that there is a higher accuracy obtained by Legendre wavelets decomposition, which is in the form of a multi-resolution analysis of the function. The solution is first found on the coarse grid points, and then it is refined by obtaining higher accuracy with help of increasing the level of wavelets. The accurate implementation of the classical numerical methods on Neumann's boundary conditions has been found to involve some difficulty. It has been shown here that the present methods can be easily implemented on Neumann's boundary conditions and the results obtained are accurate; the present methods, thus, have a clear advantage over the classical numerical methods. A distinct feature of the proposed methods is their simple applicability for a variety of boundary conditions. Numerical order of convergence of the proposed methods is calculated. The results of numerical tests show better accuracy of the proposed method based on Legendre wavelets for a variety of benchmark problems.

1. Introduction

In this paper, we propose two numerical methods based on Haar and Legendre wavelets to obtain solution of the following elliptic partial differential equations (EPDEs):

$$a_1(x,y)\frac{\partial^2 u}{\partial x^2} + a_2(x,y)\frac{\partial^2 u}{\partial y^2} + a_3(x,y)\frac{\partial u}{\partial x} + a_4(x,y)\frac{\partial u}{\partial y} + a_5(x,y)u = f(x,y), \tag{1}$$

where $a_1, a_2, \ldots, a_5$ and f are any functions of x and y or constants. The computation domain for the given EPDEs (1) is $[0,1]\times[0,1]$.

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There are several applications of EPDEs in Science and Engineering. Many physical processes can be modeled using EPDEs. Analytical solution of EPDEs, however, either does not exist or is hard to find. It is precisely due to this fact that several efficient and accurate methods have been developed for finding numerical solution of EPDEs. Recent contribution in this regard includes meshless methods [1–5], spline collocation methods [6–8], finite-difference methods [9–11], finite element method (FEM) [12], fast domain decomposition method [13], Sinc–Galerkin method [14], autocorrelation-based wavelets methods [15] etc. An alternative solution is proposed in the present paper in the form of collocation methods, which are based on Haar and Legendre wavelets for the numerical solution of general type of two-dimensional EPDEs.

Wavelets have numerous applications in approximation theory and have been extensively used in the context of numerical approximation in the relevant literature during the last two decades. A survey of some of the early works can be found in [16]. Important to note in this connection are wavelets, which have been used for numerical solutions of integral equations and numerical integration [17–19], ordinary differential equations [20,21], partial differential equations [22] and fractional partial differential equations [23]. These methods employ various types of wavelets. The examples include Daubechies [24], Battle–Lemarie [25], B-spline [20], Chebyshev [26], Legendre [27,28] and Haar wavelets [29,30,21,18]. On account of their simplicity, Haar wavelets have received the attention of many researchers. Applications of Haar wavelets for numerical approximations can be found in the following references [30–32,29,33–35]. Legendre wavelets, which are another type of wavelets, use Legendre polynomials as their basis functions. They have good interpolating properties and give better accuracy for smaller number of collocation points. This feature is addressed in the following sections. Applications of Legendre wavelets for numerical approximations can be found in the references [36–38].

The present work is a continuation of our earlier work [21,39] in which only Haar wavelets were used for the numerical solution of one-dimensional boundary-value (BV) problems. In developing the present method, the present work supplements Haar wavelets with Legendre wavelets with a view to find out the numerical solution of two-dimensional elliptic BV problems. The new approach based on Legendre wavelets has shown an improved accuracy as compared to Haar wavelets and some other numerical methods [6,3,14]. The advantages of the proposed method are as follows:

(i) Haar wavelets collocation method (HWCM) uses simple box functions. Consequently the formulation of numerical method based on Haar wavelets is straightforward and involves lesser manual labor.
(ii) Due to better approximating properties of Legendre wavelets, Legendre wavelets collocation method (LWCM) provides more accurate solutions as compared to HWCM and other well established methods.
(iii) Both HWCM and LWCM retain accuracy for the solution as well its derivatives.
(iv) The application of HWCM and LWCM to different types of boundary conditions in the context of EPDEs is easy and the results are accurate.

The organization of the rest of the paper is as follows. In Section 2, Haar wavelets and their integrals are defined. Section 3 describes Legendre wavelets and their integrals along with multi-resolution analysis. In Section 4, formulation of the method based on Haar wavelets while in Section 5, formulation of the method based on Legendre wavelets are explained for different sets of boundary conditions. In Section 6, a brief convergence analysis is given. Numerical results are reported in Section 7 and some conclusions are drawn in Section 8.

2. Haar wavelets

The scaling function for the family of Haar wavelets is defined on the interval $[0,1)$ as is given below.

$$h_1(x) = \begin{cases} 1 & \text{for} \quad x \in [0,1), \\ 0 & \text{elsewhere}. \end{cases} \tag{2}$$

The mother wavelet for Haar wavelets family is also defined on the interval $[0,1)$ and is given below.

$$h_2(x) = \begin{cases} 1 & \text{for} \quad x \in [0,\frac{1}{2}), \\ -1 & \text{for} \quad x \in [\frac{1}{2},1), \\ 0 & \text{elsewhere}. \end{cases} \tag{3}$$

All the other functions in Haar wavelet family are defined on subintervals of $[0,1)$ and are generated from $h_2(x)$ by the operations of dilation and translation. Each function in the Haar wavelets family defined for $x \in [0,1)$, except the scaling function, can be expressed as

$$h_i(x) = \begin{cases} 1 & \text{for} \quad x \in [\alpha,\beta), \\ -1 & \text{for} \quad x \in [\beta,\gamma), \\ 0 & \text{elsewhere}, \end{cases} \tag{4}$$

where

$$\alpha = \frac{k}{m}, \quad \beta = \frac{k+0.5}{m}, \quad \gamma = \frac{k+1}{m}, \quad i = 2,3,\ldots,2M. \tag{5}$$