Applied Mathematical Modelling 37 (2013) 753–761

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Approximation of $M/M/s/K$ retrial queue with nonpersistent customers

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ARTICLE INFO

Article history:
Received 19 August 2010
Received in revised form 9 February 2012
Accepted 28 February 2012
Available online 14 March 2012

Keywords:
Retrial queue
Nonpersistent customers
Persistence function

ABSTRACT

We consider the $M/M/s/K$ retrial queues in which a customer who is blocked to enter the service facility may leave the system with a probability that depends on the number of attempts of the customer to enter the service facility. Approximation formulae for the distributions of the number of customers in service facility, waiting time in the system and the number of retrials made by a customer during its waiting time are derived. Approximation results are compared with the simulation.

1. Introduction

Retrial queues are characterized by the following features: a customer who is blocked to enter the service facility upon arrival may join the virtual group of blocked customers, called orbit and repeats its requests for service after a random amount of time. Classical models of retrial queues are considered as an alternative to the queueing models with losses to represent the feature of redialing in telephone services. A feature in telephone services is that customers are impatient and do not repeat their attempts infinitely. A customer who is blocked to enter the service facility may leave the system with a probability that depends on the number of its attempts. This situation can be described in terms of persistence function $\{H_k,\ k=1,2,\ldots\}$, where H_k is the probability that after the kth attempt fails, a customer will make the $(k+1)$st one.

In this paper, we consider an $M/M/s/K$ retrial queue with impatient customers which consists of a service facility with s servers and $K-s$ waiting space and an orbit with infinite capacity. Service times of customers are independent of each other and have a common exponential distribution with parameter μ. Customers arrive from outside according to a Poisson process with rate λ. When an arriving customer finds the service facility full, the customer may join orbit and repeats its request after an exponential amount of time. The customers in orbit behave independently of each other. We assume that the retrial rate may depend on the number of failures to enter the service facility and let γ_k be the retrial rate of the customer that has experienced blocking k times. The customers are impatient and the impatience of customers is governed by the persistence function $\{H_k,\ k=1,2,\ldots\}$. For a technical reason, we assume that the number of retrials of a customer from orbit is limited by m, that is, $H_k=0$ for $k\geqslant m+1$. Since $H_k=0$ for $k\geqslant m+1$, it can be easily seen that the system is always stable.

Much effort has been spent to analyze the retrial queues with the special case of persistence function $H_1=\alpha,\ H_2=H_3=\cdots=\beta$. Closed form solutions for the system have not been obtained except for a few special cases, for example, $M/G/1/1$ retrial queue with $\alpha\leqslant 1$ and $\beta=1$ and $M/M/1/1$ retrial queue with $\alpha\leqslant 1$ and $\beta\leqslant 1$, see [1, Section 3.3]. For multiple server case, some algorithms and approximations are presented, e.g. [1–5]. For more detailed references and the related results, see Falin and Templeton [1, Chapter 5], Artalejo and Gómez-Corral [6, Chapter 3] and Wolf [7, Chapter 7]. Shin and

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http://dx.doi.org/10.1016/j.apm.2012.02.029

Choo [4] use the generalized truncation method for the stationary distribution of $M/M/s$ retrial queue in which α and β may depend on the number of customers in service facility. Recently Shin and Moon [8] propose an approximation of the $M/M/s/s$ retrial queue with the persistence function $H_j = 1,\ 1 \leqslant j \leqslant m$ and $H_j = 0$ for $j \geqslant m+1$ for a given constant m.

The objectives of this paper are to show the effectiveness of the method in [8] and to investigate the behavior of the system with more general persistence function.

The paper is organized as follows. Some preliminaries for approximations are sketched in Section 2. Approximations for the queue length distributions and waiting times are presented in Sections 3–5. In Section 6, accuracy of approximations and the effects of parameters to the performance measures are investigated numerically. Finally, conclusions are given in Section 7.

2. Exact results

We classify the customers in orbit into m different types. Let a customer in orbit who has failed k times to enter the service facility be the type k-customer, $1 \leqslant k \leqslant m$. Thus if a type k-customer is blocked again, then the customer becomes the customer of type $k+1$ with probability H_{k+1} or leaves the system with probability $\overline{H}_{k+1} = 1 - H_{k+1},\ 1 \leqslant k < m$. The customers in service facility are denoted by type 0 ones. Let $X_k^{(m)}$ be the number of customers of type $k,\ k = 0,1,\ldots,m$ in stationary state. We fix a finite number m and for simplicity, we write X_k instead of $X_k^{(m)}$ if it is not confused in the context and let $P_{kj} = P(X_k = j)$ be the distribution of $X_k,\ 1 \leqslant k \leqslant m$.

Note that the state space of the random vector $(X_0,\ldots,X_m)$ is $\mathcal{S} = \{0,1,2,\ldots,K\} \times \mathbb{Z}_+^m$, where $\mathbb{Z}_+ = \{0,1,2,\ldots\}$ is the set of nonnegative integers. Let $P(\boldsymbol{n}) = P(X_k = n_k,\ k = 0,1,\ldots,m)$ for $\boldsymbol{n} = (n_0,n_1,\ldots,n_m) \in \mathcal{S}$ and $P(\boldsymbol{n}) = 0$ otherwise. Then the balance equations are given as follows: for $\boldsymbol{n} \in \mathcal{S}$ with $0 \leqslant n_0 < K$,

$$\left(\lambda + \mu_{n_0} + \sum_{k=1}^{m} n_k\gamma_k\right)P(\boldsymbol{n}) = \lambda P(\boldsymbol{n} - \boldsymbol{e}_0) + \mu_{n_0+1}P(\boldsymbol{n} + \boldsymbol{e}_0) + \sum_{k=1}^{m}(n_k+1)\gamma_k P(\boldsymbol{n} - \boldsymbol{e}_0 + \boldsymbol{e}_k) \tag{2.1}$$

and for $\boldsymbol{n} \in \mathcal{S}$ with $n_0 = K$,

$$\begin{aligned}\left(\lambda H_1 + \mu_K + \sum_{k=1}^{m} n_k\gamma_k\right)P(\boldsymbol{n}) &= \lambda P(\boldsymbol{n} - \boldsymbol{e}_0) + \lambda H_1 P(\boldsymbol{n} - \boldsymbol{e}_1) + \sum_{k=1}^{m}(n_k+1)\gamma_k P(\boldsymbol{n} - \boldsymbol{e}_0 + \boldsymbol{e}_k)\\ &\quad + \sum_{k=1}^{m}(n_k+1)\gamma_k H_{k+1}P(\boldsymbol{n} + \boldsymbol{e}_k - \boldsymbol{e}_{k+1}) + \sum_{k=1}^{m}(n_k+1)\gamma_k \overline{H}_{k+1}P(\boldsymbol{n} + \boldsymbol{e}_k),\end{aligned} \tag{2.2}$$

where $\mu_n = \min(n,s)\mu$ and $\boldsymbol{e}_j,\ (j = 0,1,\ldots,m)$ is the $(m+1)$-dimensional vector whose jth component (beginning from 0th component) is 1 and others are all 0, and $\boldsymbol{e}_{m+1} = (0,\ldots,0) \in \mathcal{S}$.

Letting $n_0 = j$ and summing over $n_1,n_2,\ldots,n_m$ in (2.1) and (2.2), it can be seen that the distribution $P_{0j},\ j = 0,1,\ldots,K$ of X_0 satisfy the following equations

$$(\lambda_j + \mu_j)P_{0j} = \lambda_{j-1}P_{0,j-1} + \mu_{j+1}P_{0,j+1}, \quad 0 \leqslant j < K, \tag{2.3}$$

where $P_{0,-1} = 0$ and

$$\lambda_j = \lambda + \sum_{k=1}^{m}\gamma_k L_{kj}, \quad j = 0,1,\ldots,K-1, \tag{2.4}$$

with

$$L_{kj} = \mathbb{E}[X_k|X_0 = j], \quad 0 \leqslant k \leqslant m,\ 0 \leqslant j \leqslant K.$$

Thus we have from (2.3) that

$$P_{0j} = P_{00}\prod_{i=1}^{j}\left(\frac{\lambda_{i-1}}{\mu_i}\right), \quad j = 1,2,\ldots,K, \tag{2.5}$$

with

$$P_{00} = \left[1 + \sum_{j=1}^{K}\prod_{i=1}^{j}\left(\frac{\lambda_{i-1}}{\mu_i}\right)\right]^{-1}. \tag{2.6}$$

Thus the stationary distribution of the number of customers in service facility is identical to that of finite birth-and-death process with birth rates $\{\lambda_j\}_{j=0}^{K-1}$ and death rates $\{\mu_j\}_{j=1}^{K}$. The unknown parameters $\lambda_j,\ 0 \leqslant j \leqslant K-1$ will be determined in the next section.

For each $k,\ 1 \leqslant k \leqslant m$ and $n_k = j$, summing over $n_0,\ldots,n_{k-1},n_{k+1},\ldots,n_m$ in (2.1) and (2.2) and then summing over j from 0 to $j-1$, we have that for $j \geqslant 1$

$$jP_{kj} = \begin{cases} \frac{1}{\gamma_k}\lambda P(X_1 = j-1, X_0 = K)H_1, & k = 1,\\ \frac{1}{\gamma_k}\gamma_{k-1}\mathbb{E}[X_{k-1}1_{\{X_0=K\}}]H_k, & 2 \leqslant k \leqslant m,\end{cases}$$