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Optimal harvesting in a predator–prey model with Allee effect and sigmoid functional response

Alejandro Rojas-Palma, Eduardo González-Olivares *

Grupo de Ecología Matemática, Instituto de Matemáticas, Pontificia Universidad Católica de Valparaíso, Chile

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A B S T R A C T

This work deals with the determination of the optimal harvest policy in an open access fishery in which both prey and predator species are subjected to non-selective harvesting.

The model is described by autonomous ordinary differential equation systems, the functional response of the predators is Holling type III and the prey growth is affected by the Allee effect. The catch-rate functions are based on the *catch per unit effort* (CPUE) or Schaefer's hypothesis.

The problem of determining the optimal harvest policy is solved by using Pontryagin's maximal principle. The problem here studied is to maximize a cost function representing the present value of a continuous time-stream of revenue of the fishery.

1. Introduction

Bioeconomic modelling of the exploitation and management of biological resources, like fisheries and the techniques associated with the exploitation of these resources have been discussed by Clark [1] in models on single species fisheries submitted to harvest, but it is difficult to construct and study a realistic model of a multispecies interaction because the model may not be analytically tractable. It is also difficult to determine the optimal harvest policy of a n-dimensional model having more than one control variable [2,1,3].

We analyze a continuos time predator–prey model of Gause type [4] considering that

(i) The prey growth is affected by the Allee effect.
(ii) The predator is generalist.

The studied problem deals with the sustainable exploitation of two interacting species, one is the prey and the other is the predator involving combined or nonselective harvesting of the two populations considering that the fishing effort is the same for both species by using the *Schaefer's hypothesis* (CPUE) [5].

The dynamical behaviour of the exploited model has already been studied [6,7]. The optimal policy of exploitation is derived using Pontryagin's maximal principle [8].

Allee effect or *depensation* [9,10] is a situation at low population densities, where the per-individual growth rate is an increasing function of population density and it occurs whenever fitness of an individual in a small or sparse population decreases as the population size or density declines [11].

* Corresponding author.
E-mail addresses: alejandro.rojas.p@mail.ucv.cl (A. Rojas-Palma), ejgonzal@ucv.cl (E. González-Olivares).

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Fig. 1. Strong Allee effect $m > 0$.

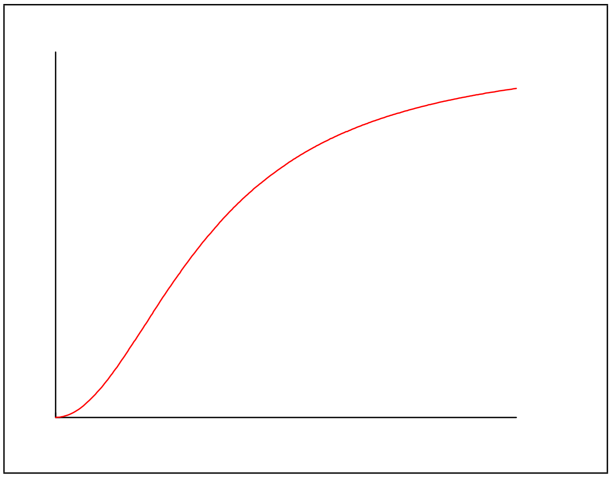

Fig. 2. Holling type III functional response.

The most common equation describing the Allee effect for a single species is given by

$$\frac{dx}{dt} = r\left(1 - \frac{x}{K}\right)(x - m)x, \tag{1}$$

If $m > 0$, the equation describes the *strong* Allee effect and m represents the minimum of viable population (see Fig. 1). The population growth rate decreases if the population size is below the threshold level m [12] and population goes to extinction. If $m \leqslant 0$, it is said that the population is affected by a *weak Allee effect* [13,24]. In fisheries the same phenomena are known as *critical* and *pure depensation*, respectively [11,12,23].

On the other hand, the *functional response of predators* or *consumption rate function* refers to the change in the density prey attacked per unit time per predator as prey density changes [4]. In most predator–prey models considered in the ecological literature, the predator response to prey density is assumed to be monotonic increasing; this inherent assumption means that the more prey population in the environment the better for the predator population [13].

In this work, we have assumed that the functional response is expressed by the function

$$h(x) = \frac{qx^2}{x^2 + a^2},$$

corresponding to Holling type III [14].

Biologically, a sigmoid functional response explains the fact that at low densities of prey population, the effect of predation is low, but if the population size increases, predation is then more intensive (see Fig. 2). This phenomenon appears in a number of interactions in the real world and in this case it is said that the predator is generalist, because if the prey population size is low, it would then seek other food alternatives [4].