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Adaptive meshless Galerkin boundary node methods for hypersingular integral equations

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ABSTRACT

Adaptive refinement techniques are developed in this paper for the meshless Galerkin boundary node method for hypersingular boundary integral equations. Two types of error estimators are derived. One is a perturbation error estimator that is formulated based on the difference between numerical solutions obtained using two consecutive nodal arrangements. The other is a projection error estimator that is formulated based on the difference between the numerical solution itself and its projection. These error estimators are proven to have an upper and a lower bound by the constant multiples of the exact error in the energy norm. A localization scheme is presented to accomodate the non-local property of hypersingular integral operators for the needed computable local error indicators. The convergence of the adaptive meshless techniques is verified theoretically. To confirm the theoretical results and to show the efficiency of the adaptive techniques, numerical examples in 2D and 3D with high singularities are provided.

1. Introduction

In many engineering problems governed by partial differential equations, singularities can occur when there have abrupt changes in the boundary conditions or on the boundary itself, or there have discontinuities in the material properties. Singularities are known to deteriorate severely the numerical accuracy and slow down the convergence rate of any standard numerical methods. If the singularity is known a priori, the form of the singularity can be properly incorporated into the numerical scheme and then a more effective method may be constructed [1,2]. However, in many cases, analysts have no a priori knowledge of the location of the areas in which such singularity will occur. An ideal computational scheme would then require the ability to automatically refine the nodes in the vicinity of the singularity, as the computation proceeds. Therefore, adaptive refinement techniques are much more flexibility in dealing with this type of problems.

Adaptive techniques have been extensively used in the field of computational science and engineering. In the past few decades, the research of adaptive techniques has seen enormous activities in the finite element method (FEM) and the boundary element method (BEM) [3,4]. Both methods depend on the generation of meshes, adapted or not. Mesh generation in some situations is still a time consuming process, arduous and fraught with pitfalls. Because the main drawback of the FEM and the BEM is caused by mesh, it motivates and leads to the development of a new class of computational methods, meshless (or meshfree) methods, which is formulated based on a set of scattered nodes [5,6].

In meshless methods, since there is no pre-specified connectivity or relationships among nodes, nodes can be conveniently inserted or removed for the refinement or coarsening procedures. This characteristic is the most desirable merit for adaptive analysis and makes meshless methods especially suited for adaptive techniques. During the last two decades, considerable efforts have been done in developing adaptive algorithms for domain type meshless methods such as the

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h–*p* meshless method [7], the element free Galerkin (EFG) method [6,8,9], the reproducing kernel particle method [10], the extended/generalized FEM [11], the finite point method [12], the point interpolation method [13] and the smoothed FEM [14].

Boundary integral equations (BIEs) and BEMs have been widely used for the numerical solution of boundary value problems. Some boundary type meshless methods have been developed by the combination of the moving least-squares (MLS) approximations with BIEs, such as the boundary node method (BNM) [4], the boundary element-free method [15] and the Galerkin boundary node method (GBNM) [16,17]. Besides, some boundary type meshless methods based on using the point interpolation method (PIM) [18] to construct shape functions have also been developed. Typical of them are the boundary PIM [6,19–21], the hybrid radial BNM [22] and the hybrid boundary PIM [23]. These PIM-based meshless methods can exactly satisfy boundary conditions, since their shape functions possess delta function property. As the BEM, all boundary type meshless methods have emerged as promising numerical techniques in scientific computing. Nonetheless, due to the nonlocal character of integral operators and the appearance of fractional-order or negative Sobolev spaces, the current state-of-technology of adaptive analysis of such meshless methods is barely at the introductory level. Chati et al. have pioneered adaptive algorithms for the BNM using hypersingular residual techniques similar to those used in the BEM [4,24]. The efficiency of these adaptive algorithms has been demonstrated numerically, but the associated mathematical foundation is not available.

The meshless GBNM combines a variational form of BIEs with MLS approximations. Unlike other MLS-based meshless methods, boundary conditions in the GBNM do not present any difficulty and can be satisfied with ease by multiplying the MLS shape function and integrating over the boundary. Another intriguing feature of the GBNM is the conservation of the symmetry and positive definiteness of the variational formulation in numerical implementation. This method has been successfully used for problems in potential theory [16,17,25], linear elasticity [26] and fluid mechanics [27]. The convergence analysis and a priori error estimate have been derived, which guarantee that this method will converge to the true solution.

Recently, the GBNM has been extended for a posteriori error estimate and adaptivity for weakly singular BIEs governing boundary value problems [28]. In that work the problem was solved twice via two consecutive nodal arrangements. The numerical solution in the second analysis is expected to be more accurate than the initial and then, a perturbation error estimator is defined as the difference between both solutions. This estimator was coupled with an *h*-refinement technique to form an effective adaptive meshless algorithm. The results in Ref. [28] were only shown for weakly singular BIEs. Since hypersingular BIEs have proven advantageous and have become a useful alternative approach in a variety of applications (e.g. wave scattering and fracture mechanics) [29], one aim of this paper is to extend these results to hypersingular BIEs.

In the perturbation error estimator, the numerical solution corresponding to the initial coarse nodal arrangement has to be computed, which is a temporary result only. Commonly, the side results should be computed with as less computational cost as possible. To avoid the direct computation of the side result, this paper further develops a projection error estimator that is achieved via calculating the difference between the GBNM solution itself and its projection. With the help of a localization scheme, computable local error indicators and efficient adaptive techniques are then established for hypersingular BIEs. Furthermore, because not much is rigorously known on the convergence of adaptive meshless techniques, the convergence proof of the developed adaptive meshless techniques is also derived.

An outline of this paper is as follows. The next section gives some notations and preliminaries to be used later. Section 3 outlines the formulations of the GBNM for hypersingular BIEs. In Section 4, a posteriori error estimators are established. Section 5 provides adaptive techniques and the associated convergence results. Numerical examples are given in Section 6. Section 7 contains some conclusions.

2. Notations and preliminaries

Throughout the paper, we use the following notations. Let Ω be a bounded or unbounded domain in $\mathbb{R}^d (d = 2, 3)$ with boundary Γ. A general point in $\mathbb{R}^d$ is denoted by $\mathbf{x} = (x_1, \ldots, x_d)$ or $\mathbf{y} = (y_1, \ldots, y_d)$.

Let $\mathcal{T}_h$ denote a subdivision of the boundary Γ into N non-overlapping cells Γ_i, then $\Gamma = \bigcup_{i=1}^{N} \Gamma_i$. Besides, we use $h \in L^\infty(\Gamma)$ to denote the local cell-size function, which can be defined as

$$h|_{\Gamma_i} := h_i := \operatorname{diam}(\Gamma_i), \Gamma_i \in \mathcal{T}_h.$$

It should be pointed out that Γ_i is not a boundary element, and therefore no shape function is dependent on it. In MLS-based boundary type meshless methods, boundary nodes can be located along cells and MLS approximations are defined in terms of nodes alone. In meshless methods, the number of nodes corresponding to each cell is arbitrary. However, to carry out accurate integration via Gaussian quadrature, it is suggested to introduce a small number of nodes per cell [4,6,7]. It has been shown in Refs. [4,17,24,27] that one node per cell and the location of the node at the centroid of the cell can yield excellent results. This nodal placement is used in this study. More specifically, one boundary node $\mathbf{x}_i$ is selected for each cell Γ_i and is located at its centroid. We then use $Q_h = \{\mathbf{x}_i\}_{i=1}^{N}$ to denote the set of these boundary nodes.

Suppose that the boundary Γ is the union of piecewise smooth segments called panels. The curvilinear coordinate on Γ is defined by $\mathbf{s}$. For any point $\mathbf{x} \in \Gamma$, assume that the influence domain of $\mathbf{x}$ is $\Re(\mathbf{x})$. The influence domain of the boundary node $\mathbf{x}_i$ is $\Re_i = \Re(\mathbf{x}_i)$. For any $\mathbf{x}(\mathbf{s}) \in \Gamma$, let $\kappa(\mathbf{s})$ be the number of boundary nodes in $\Re(\mathbf{s})$ and let $\wedge(\mathbf{s}) := \{I_1, I_2, \ldots, I_\kappa\}$ be the set of the global sequence numbers of these boundary nodes. In order to avoid the discontinuity at edges and corners, we restrict