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A numerical method for the viscous incompressible Oseen flow in shape reconstruction ☆

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ABSTRACT

This paper is concerned with the shape reconstruction of a bounded domain with a viscous incompressible fluid driven by the Oseen equations. For the approximate solution of the ill-posed and nonlinear problem we propose a regularized Gauss–Newton method. A theoretical foundation for the method is given by establishing the differentiability of the boundary value problem with respect to the boundary in the sense of the domain derivative. The results of several numerical experiments show that our theory is useful for practical purpose, and the proposed algorithm is feasible.

1. Introduction

This paper deals with the problem of the shape reconstruction for two-dimensional flows governed by the Oseen equations. This problem arises in aerospace, automotive, hydraulic, ocean, structural and wind engineering. Example applications include aerodynamic design of automotive vehicles, train, low speed aircraft and hydrodynamic design for ship hulls, turbomachinery and offshore structures.

For the shape reconstruction by the domain derivative method, many people have contributed to it. Hettlich solved the inverse obstacle scattering problem for sound soft and sound hard obstacles [1,2], Kress considered an inverse conduction scattering problem for shape and impedance in [3]. In [4,5], the three authors dealt with the inverse boundary problem for the time-dependent heat equation only in the case of perfectly conducting and insulating inclusions. In [6], we solved a shape reconstruction problem for the heat conduction with mixed condition, and we dealt with the shape reconstruction of a viscous incompressible fluid driven by the Stokes flow in [7].

This paper consists of three parts. In the remainder of the section we establish the notations that will be used throughout of the work. Section 2 is devoted to introduce Piola transformation for divergence free condition, and we derive a formulation for the derivative of the solution with respect to the boundary, which called "domain derivative". This representation is important, for it is the key to deriving many of the properties of the domain derivative and is required for the numerical analysis. In Section 3, we apply regularized Gauss–Newton method in solving the inverse problem numerically. The numerical results show that our theoretical work is correct and the method is feasible and effective in practice.

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Before closing this section, we introduce the following functional spaces which will be used throughout this paper, and they are the standard notations for Sobolev spaces (see [8]):

$$\mathbf{H}_0^1(\Omega) := \left\{ \boldsymbol{v} \in \mathbf{H}^1(\Omega),\ \boldsymbol{v}|_{\partial\Omega} = 0 \right\},$$

and

$$\mathbf{H}_0^1(\mathrm{div}, \Omega) := \left\{ \boldsymbol{v} \in \mathbf{H}_0^1(\Omega), \mathrm{div}\ \boldsymbol{v} = 0 \quad \text{in } \Omega \right\}.$$

2. The domain derivative

Let Ω be the fluid domain in $\mathbb{R}^N (N = 2 \text{ or } 3)$, and the boundary $\partial\Omega := \Gamma_s \cup \Gamma_d \cup \Gamma$ be smooth. The fluid is described by its velocity $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ and pressure p satisfying the Oseen problem:

$$\begin{cases} -\nu\Delta\boldsymbol{y} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{y} + \nabla p = \mathbf{g} & \text{in} \quad \Omega \\ \mathrm{div}\ \boldsymbol{y} = 0 & \text{in} \quad \Omega \\ \boldsymbol{y} = \boldsymbol{y}_d & \text{on} \quad \Gamma_d \\ \boldsymbol{y} = 0 & \text{on} \quad \Gamma_s \cup \Gamma, \end{cases} \tag{2.1}$$

where ν stands for the kinematic viscosity coefficient, and $\boldsymbol{w} : \Omega \to \mathbb{R}^N$ is a vectorial function such that $\mathrm{div}\boldsymbol{w} = 0$ in Ω. Let $\mathbf{g} \in \boldsymbol{L}^2(\Omega)$ be a given vector function in Ω, and $\boldsymbol{y}_d$ satisfies:

$$\int_{\Gamma_d} \boldsymbol{y}_d \cdot \boldsymbol{n}\ \mathrm{d}s = 0.$$

The Eq. (2.1) can be reduced to the homogeneous equations by setting $\boldsymbol{y} = \boldsymbol{u} + \boldsymbol{m}$ (see [9]):

$$\begin{cases} -\nu\Delta\boldsymbol{u} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in} \quad \Omega \\ \mathrm{div}\ \boldsymbol{u} = 0 & \text{in} \quad \Omega \\ \boldsymbol{u} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

where $\boldsymbol{f} = \mathbf{g} + \nu\Delta\boldsymbol{m} - (\boldsymbol{w}\cdot\nabla)\boldsymbol{m}$, and $\boldsymbol{m}$ satisfies:

$$\begin{cases} \mathrm{div}\ \boldsymbol{m} = 0 & \text{in} \quad \Omega \\ \boldsymbol{m} = \boldsymbol{y}_d & \text{on} \quad \Gamma_d \\ \boldsymbol{m} = 0 & \text{on} \quad \Gamma_s \cup \Gamma. \end{cases} \tag{2.3}$$

We define the bilinear form and the trilinear form as follows:

$$a(\boldsymbol{u}, \boldsymbol{v}) = \nu \sum_{i,j=1}^{n} \int_\Omega (D_i u_j)(D_i v_j)\mathrm{d}x, \quad \forall\ \boldsymbol{u},\ \boldsymbol{v} \in \mathbf{H}^1(\Omega),$$

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \sum_{i,j=1}^{n} \int_\Omega u_i(D_i v_j)w_j\ \mathrm{d}x, \quad \forall\ \boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w} \in \mathbf{H}^1(\Omega).$$

Continuity of the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$ can be demonstrated, these conditions guarantee the existence and uniqueness of a solution $\boldsymbol{u}$ (see [10,11]).

Taking the scalar product of (2.2) with a function $\boldsymbol{v} \in \mathbf{H}_0^1(\mathrm{div}, \Omega)$ we obtain:

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}). \tag{2.4}$$

Let a perturbation of the interior boundary Γ be specified by

$$\Gamma_h = \{\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x}),\ \boldsymbol{x} \in \Gamma\},$$

which is a C^2 boundary of a perturbed domain Ω_h, if the vector field $\boldsymbol{h} \in C^2(\Gamma)$ is sufficiently small. We choose an extension of $\boldsymbol{h} \in C^2(\Omega)$ with $\|\boldsymbol{h}\|_{C^2(\Omega)} \leqslant c\|\boldsymbol{h}\|_{C^2(\Gamma)}, c > 0$, which vanishes in the exterior of a neighborhood of Γ, and define the diffeomorphism $\varphi(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x})$ in Ω. If the inverse function of φ is denoted by ψ, J_φ, J_ψ and J_h are Jacobian matrices.

Let $\boldsymbol{u}_h \in \mathbf{H}_0^1(\mathrm{div}, \Omega_h)$ be the solution of corresponding boundary value problem:

$$\nu \int_{\Omega_h} \nabla\boldsymbol{u}_h \cdot \nabla\boldsymbol{v}_h\ \mathrm{d}x_h + \int_{\Omega_h} (\boldsymbol{w}_h \cdot \nabla)\boldsymbol{u}_h \cdot \boldsymbol{v}_h\ \mathrm{d}x_h = \int_{\Omega_h} \boldsymbol{f}_h \cdot \boldsymbol{v}_h\ \mathrm{d}x_h, \tag{2.5}$$

for all $\boldsymbol{v}_h \in \mathbf{H}_0^1(\mathrm{div}, \Omega_h)$.

It is well known that the divergence free condition coming from the fact that the fluid has an homogeneous density and evolves as an incompressible flow, and it is very difficult to impose on the mathematical and numerical point of view. Since