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Combat modelling with partial differential equations

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ABSTRACT

The limitations of the classic work of Lanchester on non-spatial ordinary differential equations for modelling combat are well known. We present work seeking to more realistically represent troop dynamics and to enable a deeper understanding of the nature of conflict. We extend Lanchesters ODEs, constructing a new physically meaningful system of partial differential equations. Spatial force movement and troop interaction components are represented with both local and non-local terms, expanding upon the swarming behaviour of fish and birds proposed by Mogilner et al. We are able to reproduce crucial behaviour such as the emergence of cohesive density profiles and troop regrouping after suffering losses in both one and two dimensions.

1. Introduction

Continuous time approaches to combat modelling have not received a great deal of attention or development since the seminal research of Lanchester [1] in 1914 when he developed the equations:

Square Law for Collective combat:

$$\frac{du}{dt} = -k_u v(t), \quad u(0) = u_0, \quad k_u > 0, \tag{1}$$

$$\frac{dv}{dt} = -k_v u(t), \quad v(0) = v_0, \quad k_v > 0. \tag{2}$$

Linear Law for Individual combat:

$$\frac{du}{dt} = -k_{vu} v(t)u(t), \quad u(0) = u_0, \quad k_{vu} > 0, \tag{3}$$

$$\frac{dv}{dt} = -k_{uv} u(t)v(t), \quad v(0) = v_0, \quad k_{uv} > 0. \tag{4}$$

Lanchester modelled force dynamics as two forces $u = u(t)$ and $v = v(t)$ with initial sizes u_0 and v_0, respectively. The constants a and b are known as *Lanchester attrition-rate coefficients* and the addition of the opposing force's density converts the Collective or Square Law equations (1) and (2) into the Individual or Linear Law equations (3) and (4). His set of ordinary differential equations have greatly influenced military decision making for many years and permeate military thought and analysis to this day. However, the underlying Command and Control structure implied by these equations is that all

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individuals have perfect knowledge of the enemy, systematically killing opponents until a winner is determined either by total annihilation or when force numbers reach a predetermined level. That is, despite variations in communications, weapon lethality, terrain effects, location in the domain, etc., individual soldiers are deemed to have equal capabilities and affect each other equally. While perhaps applicable to ancient and outdated forms of warfare, this underlying assumption must be addressed if these equations are to be applied to modern warfare that relies heavily on discrepancies of terrain, communications, sensors, weapon and soldier capabilities. Many military researchers recognise this limitation and are seeking to derive more realistic representations to enable a deeper understanding of the nature of conflict, see for example [2–4].

Progress has been made by Protopopescu et al. [5,6], Rusu [7] and Jaiswal and Nagabhushana [8], addressing some of the major criticisms of the ODEs and reflecting the development in numerical techniques and computational ability of the 1980s and 1990s. Spatial and temporal variation and local and non-local firing effects were successfully modelled using standard Method of Lines and Finite Difference techniques, and a suite of basic military manoeuvres demonstrated. Frontal attack, turning manoeuvres, envelopment, infiltration [7], reserve deployment and termination decision rules have been implemented. However, troop formation was an *artefact of initial profiles chosen*; that is, a formation initially set as a bivariate Gaussian remained roughly as such as all scenario results published were stopped after a brief period of force interaction. Furthermore, spatially dependent velocity fields resulted in unacceptable numerical losses, restricting the velocity field to a temporally dependent one. More complex velocity fields or other numerical techniques such as Finite Volume Methods have not been investigated. Recent work by Spradlin and Spradlin focused on variations of the firing terms while using either stationary forces or simple velocity terms [9].

The rise of agent-based or cellular-automaton models has received much attention in many disciplines, especially defence related research. Models such as Einstein [10], ISAAC [11], Map Aware Non-uniform Automata (MANA) [12] demonstrate a range of behaviour which appears to hint at some form of underlying structure. Each individual troop is modelled via a rule set relating to quantifiable capabilities such as fire-power, communications and also intangibles such as morale or desire to remain close to friendly forces. These rules encode the non-linearities necessary for a more realistic description of warfare. These non-linearities need to be understood in order to develop specialised tactics based on current capability, or enhance the procurement of future capability. We believe that the rapidly developing spatially and temporally discrete approach would be greatly assisted by the corresponding development of the continuous spatial and temporal approach. Indeed, Ilachinski [4], who has been instrumental in the development of ISAAC, stresses the need for research into nonlinear continuous dynamics, exploitation of analogous biological models and phase-space reconstruction techniques. Lauren compares MANA simulation results with fluid dynamic concepts or transition between laminar and turbulent states and maintenance of force profiles to viscosity [13]. If an appropriate suite of equations can be found and analysed, this may eliminate the need for extensive parametric studies and subsequent data mining in order to find those combinations of parameters that produce behaviour of interest in CA or agent-based models. Continuous models can be more transparent in terms of how parameter changes affect outcomes and thus more understandable.

2. Existing PDE models

Protopopescu et al. [5,6] extended the ODE formalism of the original Lanchester equations to partial differential equations in order to address a major criticism from the military modelling community: the inability to model the movement of forces throughout a domain or battlespace. An example of one such manoeuvre is a classic flanking movement where the main body of a force will advance directly towards the enemy while a subsection will covertly move around to the flank the enemy for a surprise attack. Avoidance of obstacles such as mountains or unnavigable terrain is also a key feature in modern warfare. Modern warfare has moved away from the traditional attritional form conducted in large open areas to smaller forces operating in difficult terrain such as the urban environment with definite spatial restrictions. A new form of warfare, manoeuvre warfare, relies on the principle of inflicting a disproportionate amount of damage to the enemy's weak points.

Protopopescu et al. consider equations of the following form over the domain $\mathbb{R}^2$. Let $u = u(x,y,t),\ v = v(x,y,t) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ represent the positive soldier densities of two opposing forces at a given position and time. Let the kernels $k_u(x,y,t),\ k_v(x,y,t) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ represent non-local interaction between the two forces over some finite domain R

$$\frac{\partial u}{\partial t} = \nabla \cdot (\mathbf{D_u} \nabla u) + \nabla \cdot (\mathbf{C_u} u) + u(a_u + b_u u + k_u * v) + d_u v + e_u, \tag{5}$$

$$\frac{\partial v}{\partial t} = \nabla \cdot (\mathbf{D_v} \nabla v) + \nabla \cdot (\mathbf{C_v} v) + v(a_v + b_v v + k_v * u) + d_v u + e_v, \tag{6}$$

where

$$(k_u * v)(x,t) = \int_R k_u(x - X, t) v_j(X,t) \partial X.$$

Taking (5) as an example, the physical interpretation of the individual terms are as follows. The first term represents diffusion of the force u. As the soldiers of force u move throughout the battlespace, they will tend to wander or move away slightly with respect to their fellow soldiers, such that the entire force distribution may diffuse over time. As maintaining formation is a critical requisite of a force's overall capability to, for example, respond to threats or move effectively to a