Applied Mathematical Modelling 35 (2011) 3036–3047

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

A cubic spline approximation and application of TAGE iterative method for the solution of two point boundary value problems with forcing function in integral form

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ARTICLE INFO

Article history:
Received 22 June 2010
Accepted 14 December 2010
Available online 22 December 2010

Keywords:
Non-uniform mesh
Integral form
Simpson's $\frac{1}{3}$rd rule
Cubic spline
TAGE method
Singularity

ABSTRACT

In this article, we report an efficient high order numerical method based on cubic spline approximation and application of alternating group explicit method for the solution of two point non-linear boundary value problems, whose forcing functions are in integral form, on a non-uniform mesh. The proposed method is applicable when the internal grid points of solution interval are odd in number. The proposed cubic spline method is also applicable to integro-differential equations having singularities. Computational results are given to demonstrate the utility of the method.

1. Introduction

During last four decades, there has been growing interest in developing and using highly accurate numerical methods based on cubic spline approximations for the solution of non-linear differential equations (see [1–5]). Albasiny and Hoskins [6,7] have obtained spline solutions by solving a set of equations with a tridiagonal matrix of coefficients. Bickley [8] has considered the use of cubic spline for solving linear two point boundary value problem. Fyfe [9] has discussed the application of deferred corrections to the method suggested by Bickley, by considering linear boundary value problem. In 1983, Jain and Aziz [10] have derived both second and fourth order cubic spline methods using uniform mesh for the numerical solution of two point non-linear boundary value problem. Later in 1984, Jain et al. [11] have discussed the highly accurate variable mesh methods for the numerical solution of two point singular perturbation problems. In the recent past, many authors (see [12–17]) have suggested various numerical methods based on cubic spline approximations for the solution of linear singular two point boundary value problem. Further recently, Mohanty et al. [18,19] have derived highly accurate cubic spline methods and discussed the application of alternating group explicit (AGE) methods for the solution of two point non-linear boundary value problems.

Now, we consider the non-linear differential equation with forcing function in integral form:

$$u'' = \phi(x, u, u') + \int_0^1 K(x, s)ds, \quad 0 < x,\ s < 1. \tag{1}$$

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doi:10.1016/j.apm.2010.12.013

The two point Dirichlet type boundary conditions are given by:

$$u(0) = \alpha_0, \quad u(1) = \alpha_1, \tag{2}$$

where α_0, α_1 are finite constants. We assume that $K(x,s)$ is a real valued function of both variables in the range $0 \leqslant x,\ s \leqslant 1$.

Let

$$I(x) = \int_0^1 K(x,s)ds \quad \text{and} \quad \phi(x,u,u') + I(x) = \psi(x,u,u').$$

Then we may re-write (1) as

$$u'' = \psi(x,u,u'), \quad 0 < x < 1. \tag{3}$$

For $0 < x < 1$ and $-\infty < u,\ v < \infty$ (where $u' = v$), we assume that

(i) $\psi(x,u,v)$ is continuous,
(ii) $\frac{\partial\psi}{\partial u}$ and $\frac{\partial\psi}{\partial v}$ exist and continuous,
(iii) $\frac{\partial\psi}{\partial u} > 0$ and $\left|\frac{\partial\psi}{\partial v}\right| \leqslant L$, where L is a constant.

These conditions assure us the existence and uniqueness of the above boundary value problem (see Keller [20]). In addition, we assume that $u(x) \in C^6[0,1]$ and $K(x,s) \in C^4[0,1]$.

To the authors knowledge, no third order variable mesh method based on cubic spline approximations for the solution of integro-differential Eq. (1) has been discussed in the literature so far. In this paper, using three variable mesh points we have discussed an efficient third order method based on cubic spline approximations for the solution of non-linear integro-differential Eq. (1) and the application of TAGE and Newton-TAGE iterative methods proposed by Evans [21,22]. In next section, we give mathematical derivation of the method in details. In Section 3, we discuss the application of TAGE and Newton-TAGE iterative methods for the solution of linear and non-linear integro-differential equation. In Section 4, we compare the computational results obtained by using the proposed iterative methods based on cubic spline approximations with the corresponding successive over relaxation (SOR) and Newton-SOR iterative methods (see [23–27]). Concluding remarks are given in Section 5.

2. Mathematical derivation of the method

We discretize the solution region [0,1] with the non-uniform mesh such that $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$. Our method consists of three grid points x_k, x_{k+1} and x_{k-1}, where $x_k - x_{k-1} = h_k$ and $x_{k+1} - x_k = h_{k+1}$. Grid points are given by $x_i = x_0 + \sum_{k=1}^{i} h_k,\ i = 1,2,\ldots,N+1$. The mesh ratio is $\sigma_k = h_{k+1}/h_k$. When $\sigma_k = 1$, then it reduces to the constant mesh case. Let the exact solution of $u(x)$ at the grid point x_k be denoted by $U_k = u(x_k)$ and u_k be the approximate value of U_k.

Throughout our discussion, we consider N as odd, i.e. our solution region contains odd number of internal grid point. Let us construct a numerical method for evaluating the integral $\int_0^1 G(x)dx$.

$$\text{Let} \quad \int_{x_{k-1}}^{x_{k+1}} G(x)dx = b_{-1}G_{k-1} + b_0 G_k + b_1 G_{k+1}, \tag{4}$$

where b_{-1}, b_0, b_1 are parameters to be determined and at the grid point x_k, we denote $G_k = G(x_k)$.

Further, we may write

$$G_{k-1} = G_k - h_k G'_k + \frac{h_k^2}{2}G''_k - \frac{h_k^3}{6}G'''_k + \cdots, \tag{5a}$$

$$G_{k+1} = G_k + h_{k+1} G'_k + \frac{h_{k+1}^2}{2}G''_k + \frac{h_{k+1}^3}{6}G'''_k + \cdots, \tag{5b}$$

$$\int_{x_{k-1}}^{x_{k+1}} G(x)dx = (h_k + h_{k+1})G_k + \frac{h_{k+1}^2 - h_k^2}{2}G'_k + \frac{h_{k+1}^3 + h_k^3}{6}G''_k + \frac{h_{k+1}^4 - h_k^4}{24}G'''_k + \cdots \tag{5c}$$

By the help of (5a)–(5c), comparing both sides of (4), we get

$$b_0 + b_1 + b_{-1} = h_k + h_{k+1}, \tag{6a}$$

$$b_1 h_{k+1} - b_{-1} h_k = \frac{h_{k+1}^2 - h_k^2}{2}, \tag{6b}$$

$$b_1 h_{k+1}^2 + b_{-1} h_k^2 = \frac{h_{k+1}^3 + h_k^3}{3}. \tag{6c}$$