Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematical Modelling 32 (2008) 240–247
APPLIED MATHEMATICAL MODELLING
www.elsevier.com/locate/apm

Traveling wave solutions of a nonlinear reaction–diffusion–chemotaxis model for bacterial pattern formation

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Received 1 May 2005; received in revised form 1 June 2005; accepted 22 November 2006
Available online 23 January 2007

Abstract

In this paper we consider a nonlinear reaction–diffusion–chemotaxis model for the description of the spatiotemporal evolution of the bacteria of the type *Paenibacillus dendritiformis* on a thin layer of agar in a Petri dish. We perform a traveling wave analysis for the model equation showing the existence of traveling wave solutions, in particular, the sharp wave front type solutions with minimum speed. Further, we present numerical investigations for a special case. The minimum speed is estimated and the profile of the traveling wave solution is calculated and compared for different numerical methods.

MSC: 35K57; 35A40; 37C45

Keywords: Reaction; Diffusion; Chemotaxis; Bacterial pattern formation; Traveling waves

1. Introduction

Many biological systems exhibit the phenomenon of self-organization, whereby complex spatiotemporal patterns emerge from the biological processes occurring within the system. One of these systems is the spatiotemporal patterning behavior in bacterial colonies (see [1–8]) whose study has important implications in biotechnology and related areas.

In this paper we consider a mathematical model for studying pattern formation by bacterial colonies exemplified by the growth of bacteria of the type *Paenibacillus dendritiformis* on a thin layer of agar in a Petri dish [9]. These bacteria cannot move on a dry surface, and produce a layer of lubricating fluid in which they swim. In a uniform layer of liquid, bacterial swimming is a random process which can be approximated by diffusion. The lubrication fluid flows by convection caused by motion of the bacteria and diffusion. The availability of

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doi:10.1016/j.apm.2006.11.013

nutrients affects the reproduction of bacteria, the production of lubrication fluid, and the withdrawal of bacteria into a pre-pore state. The bacteria consume the nutrients. A continuum approach to the dynamics leads to a model comprising three coupled reaction–diffusion equations with unknowns: the density of the bacteria u, the height of the lubrication layer w, and the available nutrient n, respectively. Considering the density of bacteria in the pre-pore state optionally give rise to a forth equation. Under simplifying assumptions, the variable w can be eliminated. This leads [9] to the equation for the bacterial field

$$\partial_t u = \nabla \cdot (\alpha u^k \nabla u) + R,$$

where ∇ denotes the standard differential operator in the two space dimension, and R the nett effect of reproduction and withdrawal of bacteria. The term R vanishes when $u = 0$, and depends monotonically on n in such a way that it is a source when n exceeds some critical nutrient level and a sink when n falls below this level. When chemotaxis is taken into account, the equation becomes

$$\partial_t u = \nabla \cdot (\alpha u^k \nabla u - \beta_0 u^{k+1} \chi(z) \nabla z) + R,$$

where z denotes the concentration of the chemical responsible for the chemotaxis, $\chi(z)\nabla z$ the chemical gradient sensed by the bacteria, and β_0 a constant which is positive for attraction and negative for repulsion [9]. Assuming that the motion is in one-dimension, the chemical gradient is uniform, and $R := ru(1 - u^k)$ give rise to equation [10]

$$\partial_t u = \partial_x (\alpha u^k \partial_x u) - \beta u^k \partial_x u + ru(1 - u^k).$$

This reaction term embodies the properties described previously under the simplifying assumption that the nutrient level is appropriately related to the bacterial density. The value of the bacterial density corresponding to the critical nutrient level is normalized to $u = 1$.

The purpose of this paper is to examine analytically and numerically the traveling wave problem for the following version of the model equation

$$\partial_t u = \partial_x (u^k \partial_x u) - \beta u \partial_x u + u(1 - u^q), \tag{1}$$

where $k, q > 0$ and $\beta \geqslant 0$ are constants. In Section 2, we perform the traveling wave analysis showing the existence of sharp and non-sharp traveling wave solutions. In Section 3, we present numerical computations for a special case. Section 4 contains conclusion.

2. Traveling wave analysis

We look for traveling wave solutions $u(x, t) = u(z)$ of (1), with $z = x - ct$ where $c > 0$ is a traveling wave speed. Thus, if a traveling wave solution $u(z)$ of (1) exists, then it must satisfy the following second order ODE

$$-c d_z u = d_z (u^k d_z u) - \beta u d_z u + u(1 - u^q) \tag{2}$$

and the boundary conditions

$$u(-\infty) = 1 \quad \text{and} \quad u(+\infty) = 0. \tag{3}$$

with $0 \leqslant u(z) \leqslant 1 \ \forall z \in (-\infty, +\infty)$.

To study the existence of traveling wave solutions for (1) we will analyze the corresponding phase trajectory equation of the ODE system associated with (2). Rewriting (2) as follows:

$$-c u^{k-1} d_z u = u^k d_z (u^{k-1} d_z u) + (u^{k-1} d_z u)^2 - \beta u u^{k-1} d_z u + u^k (1 - u^q)$$

and setting

$$v = -u^{k-1} d_z u, \tag{4}$$

we have the following system of ODE:

$$\begin{aligned} d_z u &= -u^{1-k} v, \\ u^k d_z v &= (-c + v)v + \beta u v + u^k (1 - u^q). \end{aligned} \tag{5}$$