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A low-order meshless model for multidimensional heat conduction problems

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ABSTRACT

In this paper, a perturbation method is used to solve a two-dimensional unsteady heat conduction problem. Low-order transfer functions are defined. Step responses are obtained and compared to the complete numerical solutions given by a meshless method. The analytical results are found to be in good agreement with numerical solutions which reveals the effectiveness and convenience of the used method.

1. Introduction

Multidimensional unsteady diffusion type parabolic problems arise in various transport processes. Closed form analytical solutions are generally difficult to obtain except in some simple situations. The need for analysis of more complicated practical situations demands the use of computational methods which allow the determination of solution with good accuracy. When the geometry is complex, finite element [1–5] or meshless [6–12] methods can be used for the complete solution of the problems at hand. Explicit, implicit or even semi-implicit time marching procedures allow the determination of the solution at each time step. Despite its advantages, a general numerical code can however present some drawbacks. In particular, it may require a hudge computational time to describe the specific details of the unsteady solution especially when a large number of discretization nodes is necessary. Further, the numerical code can not be coupled as easily to a control strategy or to solve an inverse problem for example. It is therefore useful to develop more simple models even if not fully analytical. Toward this end, a perturbation method has been proposed [13] to solve the one-dimensional heat conduction problem in planar [14], cylindrical [15] and spherical geometries [16]. It has been shown that second order models can be sufficiently accurate. This perturbation method is similar in some situations to the Adomian decomposition method [17]. The main goal of this paper is to extend the perturbation method to the case of a two-dimensional problem of complex geometry. In the following sections, we first present the method and the considered problem. Low-order transfer functions, impulse and step responses are then developed. The obtained step responses are compared to the full numerical solution which has been obtained by a meshless numerical method and an implicit in time scheme. Finally, we show that the Duhamel theorem can be employed to find the low-order responses to more complex excitations.

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2. Perturbation method

Let us first recall the basic principles of the perturbation decomposition method for solving the heat equation. Consider the general heat conduction equation:

$$\rho c \frac{\partial T}{\partial t} = div(\lambda \vec{grad}\, T), \tag{1}$$

where ρ, c and λ are the density, specific heat and heat conductivity, respectively.

The solution of Eq. (1) is sought on the domain of Fig. 1 under the following boundary conditions:

$$-\lambda \frac{\partial T}{\partial n} = h(T - f) \quad \text{on } \Gamma_1, \tag{2 - a}$$

$$T = 0 \quad \text{on } \Gamma_2. \tag{2-b}$$

In order to develop a low-order model, we introduce as in [13] the perturbation parameter ε (which will be set to one later) in the left hand side of Eq. (1) as follows:

$$\varepsilon \rho c \frac{\partial T}{\partial t} = div(\lambda \vec{grad}\, T). \tag{3}$$

We now seek the solution in the following form:

$$\theta(x, y, t) = \sum_{n=0}^{\infty} \varepsilon^n \psi_n(x, y) \cdot F_n(t), \tag{4}$$

where $\psi_n(x, y)$ and $F_n(t)$ designate the spatial and time functions at the nth order perturbation.

Substituting Eq. (4) into Eq. (3) yields an asymptotic form and the following recurrence relations:

$$div(\lambda \vec{grad}\, \Psi_0) = 0, \tag{5}$$

$$\rho c \Psi_{n-1} = div(\lambda\ \vec{grad}\, \Psi_n), \quad n > 0, \tag{6}$$

$$F_n(\tau) = \frac{dF_{n-1}}{dt}, \quad n > 0. \tag{7}$$

The boundary conditions must be verified whatever the value of ε. We therefore must have:

$$-\lambda \frac{\partial \Psi_0}{\partial n} = h(\Psi_0 - 1) \quad \text{on } \Gamma_1, \tag{8}$$

$$\Psi_0 = 0 \quad \text{on } \Gamma_2, \tag{9}$$

$$F_0(t) = f(t), \tag{10}$$

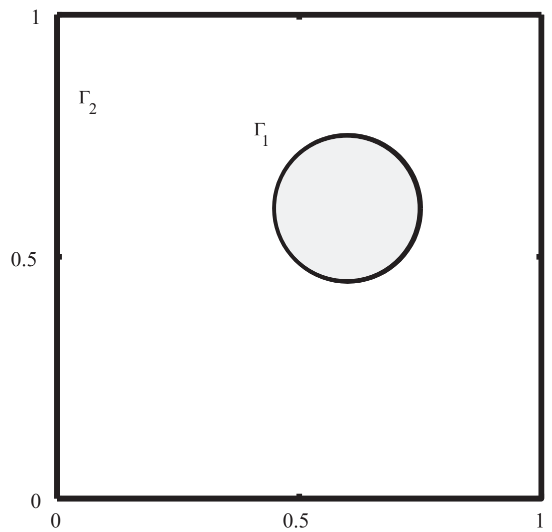

Fig. 1. Geometry of the problem. The hole is centred at $(X, Y) = (0.6L, 0.6L)$ and its diameter is $0.3L$.