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Short communication

Steady flow of incompressible fluid between two co-rotating disks

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ABSTRACT

This article provides an analytical solution of the Navier–Stokes equations for the case of the steady flow of an incompressible fluid between two uniformly co-rotating disks. The solution is derived from the asymptotical evolution of unknown components of velocity and pressure in a radial direction – in contrast to the Briter–Pohlhausen analytical solution, which is supported by simplified Navier–Stokes equations. The obtained infinite system of ordinary differential equations forms recurrent relations from which unknown functions can be calculated successively. The first and second approximations of solution are solved analytically and the third and fourth approximations of solutions are solved numerically. The numerical example demonstrates agreements with results obtained by other authors using different methods.

1. Introduction

The object of investigation of this paper is the steady flow of viscous fluid between two parallel co-rotating disks where the fluid enters an inner cylinder in a radial direction and emerges at the outer cylinder (Fig. 1). Note that the problem differs from the celebrated von Karman problem and its generalization since this problem investigates the swirling flow induced by the rotation of infinite disks [1].

It seems that the problem – in the context of application in the design of centrifugal pumps – was first studied in 1962 by Breiter and Pohlhausen [2]. From the linearized boundary layer approximations of Navier–Stokes equations they derived the analytical expressions for velocity components and pressure showing that the solution depends on kinematic viscosity, angular velocity and the distance between the disks. They also provide a numerical solution of the non-linearized equations using the finite-difference method with constant inlet profile. This line of research was continued by Rice and coworkers, whose main goal was the prediction of the performance of a centrifugal pump/compressor. They used different methods to obtain velocity and pressure distribution of flow between two disks. Thus Rice [3] studied the flow with equations derived by using hydraulic treatment of bulk flow; Boyd and Rice [4] used the finite difference method to calculate velocity and pressure for various parabolically distributed inlet velocities; and Boyack and Rice [5] used what they called the integral method, in which the velocity components are represented by a polynomial of the axial coordinate.

Another line of research of the so called laminar source-sink flow in a rotating cylindrical cavity originated with the analytical study by Hide [6], who gave the approximate asymptotic expressions for velocity components using the boundary-layer technique. Numerically, by using the finite-difference method, the problem was solved by Bennetts and Jackson [7]. Owen et al. [8] used the integral-momentum of von Karman that extended Hide's linear approximation to the turbulent flow. Recently, the steady flow between rotating disks was included in the study by Crespo del Arco et al. [9] using a pseudo-spectral numerical method.

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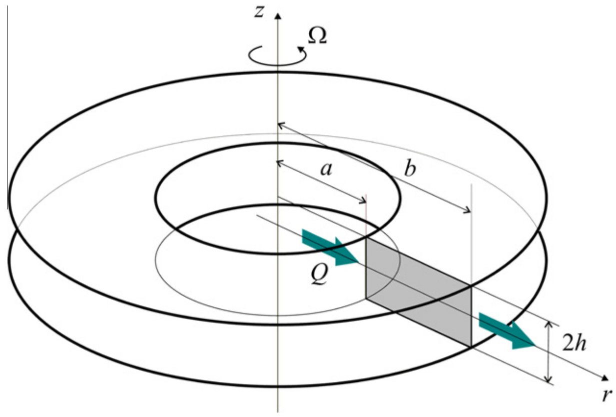

Fig. 1. Geometry of the problem.

From the above brief review of literature it is clear that the problem is analytically and especially numerically well studied and the results of calculations are in agreement with experiments. However, all available analytical solutions are based on variants of approximation. In this article, an alternative analytical solution will be presented, which is based on the asymptotic expansion of unknown functions by a method similar to those of Savage [10], who considered the stationary radial directed flow between steady disks. The article is organized as follows. After the basic equations are established, their dimensionless forms are provided. The equations are solved and the results are compared with other methods.

2. Basic equations

Consider the steady axisymmetrical isothermal flow of incompressible viscous fluid between two co-rotating disks in the absence of body force. The disks have inner radius a and outer radius b. The distance between disks is $2h$. Both disks rotate in the same direction with constant angular velocity Ω. For description of flow, the reference frame rotating with angular velocity Ω is used. In this frame, by using the cylindrical coordinate system with coordinates r and z, the continuity equation and Navier–Stokes equations have the form (cf. [11,12]):

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$\begin{aligned}
&u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}-2\Omega u=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right),\\
&u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}-\frac{uv}{r}+2\Omega v=+\nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\right),\\
&u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right),
\end{aligned} \tag{2}$$

where $u(r,z)$, $v(r,z)$, $w(r,z)$ are the components of relative velocity in radial, tangential and axial direction, $p(r,z)$ is the reduced pressure, ρ is the density, and ν is the kinematic viscosity.

Eqs. (1) and (2) are to be solved on the domain $r \in [a,b]$ and $z \in [-h,h]$ subject to the following boundary conditions along the disks' plane:

$$u(r,\pm h)=v(r,\pm h)=w(r,\pm h)=0. \tag{3}$$

The boundary condition in entrance and outer cross section requires prescribed velocity components as functions of coordinate z. Since the asymptotic series solution, which will be used, does not offer enough free parameters to satisfy this boundary condition, it is replaced by prescribing the volume flow rate Q. Therefore, at outer cross section $r = b$ one has the condition

$$Q=2\pi b\int_{-h}^{h}u(b,z)dz. \tag{4}$$

Because (4) does not refer to the interval $[a,b]$ its limits become artificial. So in this context b will be used as the reference radius. Also, because the boundary condition at the inner and outer cross sections will not be precise, the solution will not cover the inner source region and outer sink layer [8].