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Algebraic Dirichlet-to-Neumann mapping for linear elasticity problems with extreme contrasts in the coefficients

Frédéric Magoulès [a,*], François-Xavier Roux [b], Laurent Series [b]

[a] *Institut Elie Cartan de Nancy, Université Henri Poincaré, BP 239, 54506 Vandoeuvre-les-Nancy Cedex, France*
[b] *High Performance Computing Unit, ONERA, 29 av. de la Division Leclerc, BP 72, 92322 Châtillon Cedex, France*

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Abstract

The convergence of iterative based domain decomposition methods is linked with the absorbing boundary conditions defined on the interface between the sub-domains. For linear elasticity problems, the optimal absorbing boundary conditions are associated with non-local Dirichlet-to-Neumann maps. Most of the methods to approximate these non-local maps are based on a continuous analysis. In this paper, an original algebraic technique based on the computation of local Dirichlet-to-Neumann maps is investigated. Numerical experiments are presented for linear elasticity problems with extreme contrasts in the coefficients.

Keywords: Dirichlet-to-Neumann; Domain decomposition; Discontinuous coefficients; Heterogeneous media; Linear elasticity

1. Introduction

Domain decomposition methods with Lagrange multipliers have been shown to be very efficient for the parallel solution of large scale problems [1–4]. These methods are based on the partitioning of the global domain into non-overlapping sub-domains. The continuity is enforced by using additional (one or two) Lagrange multipliers [5] across the interfaces between the sub-domains. Additional augmented matrices defined on the interface are used in order to avoid possible non-well-posed sub-problems [6–8]. The definition of theses augmented matrices is linked with the absorbing boundary conditions defined on the interface between the sub-domains [9,10]. It can be shown that using some absorbing boundary conditions involving a non-local Dirichlet-to-Neumann (DtN) map of the outside of each sub-domain, leads to the optimal convergence of the iterative domain decomposition algorithm based on two Lagrange multipliers [11,12]. Several approaches have been investigated to define efficient local approximation of this DtN map. These approximations can be classified

* Corresponding author. Tel.: +33 383 684564; fax: +33 3 83 684534.
E-mail addresses: frederic.magoules@iecn.u-nancy.fr (F. Magoulès), francois-xavier.roux@onera.fr (F.-X. Roux), laurent.series@onera.fr (L. Series).

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into two major classes. The first class consists of continuous approximations of the continuous DtN operator with Fourier analysis tools [13–17]. The second class consists of discrete approximations of the discrete DtN operator with algebraic techniques [18,12,19]. In this paper, an original technique is introduced. This technique consists to define several local DtN maps and to assembly these local DtN maps in order obtain an efficient and robust approximation of the non-local DtN map.

This paper is organized as follows: Section 2 introduces the equations of linear elasticity and the linear system of equations considered. Section 3 presents the domain decomposition method with two Lagrange multipliers and two augmented matrices defined on the interface between the sub-domains. The non-local Dirichlet-to-Neumann map involved for the optimal convergence of this domain decomposition method is then briefly reminded. Section 4 introduces an original technique to approximate this non-local Dirichlet-to-Neumann map. The analogy of this original technique with a finite element method is then discussed. An asymptotic analysis of this technique is presented in Section 5, followed by numerical experiments applied to linear elasticity problems with extreme contrasts in the coefficients. Finally, Section 6 contains the conclusions of this paper.

2. Mathematical formulation

Here, the equations of linear elasticity are considered. Let an elastic body which occupies in the absence of force the set $\Omega \in \mathbb{R}^3$ with a regular boundary $\partial\Omega$. Let $u_i\,(x,t)$ denotes the displacement at position x ($x \in \mathbb{R}^3$). The symmetric strain tensor (Green tensor) ϵ_{ij} is defined by

$$\epsilon_{ij} = \frac{1}{2}(\partial_{x_j}u_i + \partial_{x_i}u_j + \partial_{x_i}u_k\partial_{x_j}u_k).$$

Assuming small displacements $\partial_{x_i}u_j \ll 1$ and small strain $\partial_{x_i}u_i \ll 1$, the linear expression of the strain tensor reduces to

$$\epsilon_{ij} = \frac{1}{2}(\partial_{x_j}u_i + \partial_{x_i}u_j).$$

In the case of an hyper-elastic body, the strain and stress tensor are linked together with the relationship:

$$\sigma_{ij} = g(\epsilon_{kl}).$$

In the following analysis, the case of a linear body is considered which means that the stress tensor is proportional to the strain tensor:

$$\sigma_{ij} = C_{ijkl}\ \epsilon_{kl}.$$

Finally, in the case of an isotropic body the Hooke law gives

$$\sigma_{ij} = \lambda\ \epsilon_{kk}\delta_{ij} + 2\mu\ \epsilon_{ij},$$

where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \text{ and } \mu = \frac{E}{2(1+\nu)},$$

denote the Lamé coefficients, E the Young modulus and ν the Poisson ratio. With these notations, applying the Hamilton principle to the continuous system of equations and assuming small displacements, the system of equations of linear elasticity can be obtained:

$$-(\lambda+\mu)\nabla(\mathrm{div}(u)) - \mu\Delta u = f, \tag{2.1}$$

where f denotes the right-hand side. The variational formulation of this system of equations gives find $u \in H^1(\Omega)$ such as $\forall v \in H^1(\Omega)$

$$a(u,v) = l(v),$$