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# Free vibration multiquadric boundary elements applied to plane elasticity

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ABSTRACT

This paper presents a new boundary element formulation for free vibration analysis of two-dimensional elastic structures. The dual reciprocity method (DRM) is introduced using the multiquadric radial basis functions (MQ). The required particular solution kernels for displacement and traction are derived and smoothed using simple mathematical trick; hence the limiting values of these kernels are computed. The eigen-problem of displacement is formulated and solved to obtain the required frequencies of different vibration modes. Finally, several numerical problems are studied to demonstrate the validity and accuracy of the developed formulation. The results are compared to those obtained from other interpolation functions, and finite element analysis to demonstrate the validity and superiority of the present formulation.



## 1. Introduction

Free vibration analysis is one of the main branches in structural dynamics. The boundary element method (BEM) [1] is one of the most efficient numerical methods for studying structures under free vibrations. The main advantage of the BEM is the need of boundary only discretization, which leads to solve complex structures in less time and with high accuracy. There are many BEM formulations for treating structures under free vibrations [2]. These formulations include the time domain, the Laplace transform, and the domain integral techniques. The main problems of these formulations are the mathematical complications founded in the first two and due to the required domain integrations in the last one. To overcome these drawbacks, Nardini and Brebbia [3] developed a new formulation for transient problems, which is named the dual reciprocity method (DRM). In the DRM, the domain integral due to inertia forces is transformed to the boundary. Such transformation process is carried out using a new collocation scheme for representing the field accelerations, or consequently the field displacements, by means of equivalent functions. Nardini and Brebbia [3] suggested the using of the conical radial basis functions (RBF), $(1 + R)$ where $R$ is interpolation distance, for approximating the new schemes for displacements and tractions.

Many researchers used the DRM to solve different applications such as potential, fluid mechanics, and heat transfer problems. A survey for such applications is given by Partridge et al. [4]. Another applications concerning transient problems have also been studied by other researchers, such as Agnantiaris et al. [5] who made a comparison between the behaviors of polynomial RBF against thin plate splines (TPS), $(R^2 \ln R)$, in elastodynamic problem. Rashed [6] used the Gaussian function to solve transient problem then he extended his formulation to the compact support functions in Ref. [7]. The non-linear applications are also studied in Refs. [8,9].

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The free vibration analysis using the DRM was considered by Nadini and Brebbia [10], which used the concept of static condensation to contribute the effect of support tractions into the final eigen-problem form. Then, free vibration analysis of different structures- whatever it is in-plane or transverse vibrations- are investigated by many researchers using different global RBFs. Brebbia and Nadini [11] suggested the adding of the collocation point coordinate value to the $R$ function to be $(X_k + R)$. Bridges [12] considered the augmented thin plate spline function (ATPS) in solving 2D in-plane vibrations. He [12] made a comparison between his results and those obtained by using $(1 + R)$ and concluded that the ATPS produces better results than the $(1 + R)$ does. Agnantiaris et al. [13] presented a general formulation for the fictitious displacements and tractions, which relates them to any assumed RBF in 3D elastic solids, and then they used the $(1 + R)$ function in studying 3D elastodynamics problems. Wilson et al. [14] used the polynomial functions in solving 3D solids. Mehraeen and Noorzad [15] introduced a RBF in the form $(c_1 + c_2\ R + c_3\ R^2 \ln R)$ and implemented it in 2D elastodynamics problems.

Selection of suitable approximation function becomes very essential in the DRM since it is a major parameter that influences the result accuracy. Several studies were carried out in order to rank the radial basis functions according to their best interpolation of multivariate data or functions. An interesting study in this field was introduced by Franke [16], who reviewed all available radial basis functions for interpolating scattered data sets. Among the tested functions, Hardy's Multiquadrics (MQ), $(R^2 + C^2)^{1/2}$, were ranked the best in accuracy, followed by Dunchon's thin plate splines (TPS). The reason for considering the MQ function to be the best in interpolations is for its exponential convergence rate [17], while this rate in the case of the TPS function is just linear [18].

A few number of researches founded using the MQ function in the context of DR/BEM. Golberg et al. [19] utilized the MQ function in approximating the forcing term of Poisson's differential equation. Then, they [19] were able to solve potential problems in the Laplacian form and concluded that the obtained results using the MQ interpolation are highly accurate. Later, Agnantiaris et al. [20] used the MQ function to investigate the behavior of 3D non-axisymmetric and axisymmetric structures under the effect of free vibrations. However, the application of MQ function in 2D elastodynamics has never been reported previously.

In this paper, the MQ radial basis function is implemented for analyzing 2D structures under free vibrations. The expressions for displacement and traction particular solutions for the MQ interpolation are derived. The limiting case (as $R \to 0$) is studied and the values of particular solutions at this case are obtained in explicit forms. The DRM eigenvalue problem is then established and solved to obtain the vibration modes and corresponding frequencies. Several examples are tested to demonstrate the accuracy and validity of the suggested formulation.

## 2. Dynamic integral equation of 2D elasticity

Consider a 2D structure with domain $\Omega$ and boundary $\Gamma$ to be subjected to in-plane dynamic loading. The general dynamic integral equation in terms of boundary displacements and tractions is given by [1]:

$$c_{ij}(\xi)u_j(\xi) + \int_{\Gamma(x)} T_{ij}(\xi,x)u_j(x)\mathrm{d}\Gamma(x) = \int_{\Gamma(x)} U_{ij}(\xi,x)t_j(x)\mathrm{d}\Gamma(x) - \rho\int_{\Omega(x)} U_{ij}(\xi,x)\ddot{u}_j(x)\mathrm{d}\Omega(x) \tag{1}$$

where $\xi$ is a source point, $x$ is a field point, $u_j(x)$ and $t_j\,(x)$ are the boundary displacement and traction, respectively at the field point, and $c_{ij}\,(\xi)$ is the jump term; $c_{ij}\,(\xi) = \delta_{ij}/2$ if $\xi \in \Gamma$ (smooth boundary) and $c_{ij}\,(\xi) = \delta_{ij}$ if $\xi \in \Omega \delta_{ij}$ is the unity matrix which equals zero when $i \neq j$, and equals one when $i = j$). $\rho$ is the mass density, and $\ddot{u}_i(x)$ is the acceleration at $x$ (the over dots denotes derivatives with respect to time). The kernels $U_{ij}(\xi,\ x)$ and $T_{ij}(\xi,\ x)$ are the two-point Kelvin fundamental solutions which are given by [2]:

$$U_{ij}(\xi,x) = \frac{1}{8\pi G(1-v)}[-(3-4v)\ln r\delta_{ij} + r_{,i}r_{,j}] \tag{2a}$$

$$T_{ij}(\xi,x) = -\frac{1}{4\pi(1-v)r}[(1-2v)\delta_{ij} + 2r_{,i}r_{,j})r_{,n} + (1-2v)(n_i r_{,j} - n_j r_{,i})] \tag{2b}$$

in which the tensor (indicial) notation is used; the comma denotes derivatives with respect to the spatial coordinates, and the index repetition means summation. $v$ is the Poisson's ratio, $G$ is the shear modulus which is given by: $G = E/2(1+v)$, in which $E$ is the modulus of elasticity, $r \equiv r\,(\xi, x)$ denotes the distance between the source and field points $(\xi, x \in \Gamma)$, $n_i$ denotes the component of the normal at the field point, and $r_{,n} = r_{,i}\ n_i$.

To eliminate the domain integral in Eq. (1), an interpolation scheme for inertia body forces is used to transfer this integral to the boundary via the DRM. Then, the final integral equation can be written as follows (for more details, see Ref. [2]):

$$c_{ij}(\xi)u_j(\xi) + \int_{\Gamma_{(X)}} T_{ij}(\xi,x)u_j(x)\mathrm{d}\Gamma(x) = \int_{\Gamma_{(X)}} U_{ij(\xi,x)t_j(x)}\mathrm{d}\Gamma(x) - \sum_{k=1}^{m}\left[c_{ij}(\xi)\Psi_{jl}(\xi,y_k) + \int_{\Gamma_{(X)}} T_{ij}(\xi,x)\Psi_{jl}(x,y_k)\mathrm{d}\Gamma(x) - \int_{\Gamma_{(X)}} U_{ij}(\xi,x)\eta_{jl}(x,y_k)\mathrm{d}\Gamma(x)\right] \times \rho\sum_{k=1}^{m} f^{-1}(x,y_k)\,\ddot{u}_l(x) \tag{3}$$

where $f(x, y_k)$ is a suitable interpolation function which will be selected later, and $\Psi_{jl}(x, y_k)$ and $\eta_{jl}(x, y_k)$ are the displacement and traction kernels, respectively between the field points $x$ and the new set of field points $y_k(k = 1 \to m)$. Such new kernels will be derived in the next section.