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DEA models for ratio data: Convexity consideration

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Abstract

Data envelopment analysis (DEA) is defined based on observed units and by finding the distance of each unit to the border of estimated production possibility set (PPS). The convexity is one of the underlying assumptions of the PPS. This paper shows some difficulties of using standard DEA models in the presence of input-ratios and/or output-ratios. The paper defines a new convexity assumption when data includes a ratio variable. Then it proposes a series of modified DEA models which are capable to rectify this problem.

Keywords: Linear programming; Data envelopment analysis (DEA); Production possibility set (PPS); Ratio data; Convexity

1. Introduction

Data envelopment analysis (DEA) is a non-parametric method for measuring efficiency of a set of Decision Making Units (DMUs) such as firms or a public sector agencies, first introduced into the operations research and management science literature by Charnes, Cooper, and Rhodes (CCR) [1]. The original CCR model was applicable only to technologies characterized by constant returns to scale globally. In what turned out to be a major breakthrough, Banker, Charnes, and Cooper (BCC) [2], extended the CCR model to accommodate technologies that exhibit variable returns to scale. Both CCR and BCC have been spread to many topics and with many successful applications and case studies as reported in the DEA literature [3,4].

One of the main assumptions in the definition of efficiency measure underlying DEA is the convexity axiom. Hence many researchers have paid specific attention to the convexity assumption. For example the convex projection approach of Petersen [5], Bogetoft [6], and Bogetoft et al. [7] all dispensed with the assumption of global convexity while at the same time presuming convexity of input and output sets. The 'convex pairs' approach of Agrell and Bogetoft [8] dispenses with the input sets and output sets that may themselves be non-convex unions of convex subsets. This approach allows modeling of overall non-convex technologies while retaining convexity in the input and output dimensions in a local sense while Podinovski [9] extended their model in which each input and output is treated individually with respect to the convexity assumption.

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Other attempts have been made to relax the standard convexity assumption in DEA including Chang [10], Post [11] and Dekker and Post [12]; who have considered the cases where the input isoquants or the output isoquants but not both are assumed convex. Kuosmanen [13] and Kuosmanen [14] replaced the convexity property of the PPS by conditional convexity. Post [11] considers a convex transformation of a non-convex possibility set by means of so-called trans-convex functions.

DEA analysis is usually undertaken with absolute numerical data, which among other things reflect the size of the units. There are some cases reported in the literature [3] that the authors used ratio variables rather than absolute numbers as input (input-ratio) and/or output (output-ratio). For example, a health database presents almost all its indicators in the form of ratios, such as expenditure as a percentage of GDP, discharge rates from hospital per 100,000 population, and annual number of days lost through sickness per employee [15]. Hollingsworth and Smith [15] explained that the CCR formulation of Charnes, Cooper and Rhodes [1] should not be used; instead they have shown that BCC formulation of Banker, Charnes and Cooper [2] must be applied when input and/or output include a ratio variable. This paper highlights the convexity problem in the standard Production Possibility Set (PPS) and draws alternative DEA models in the presences of output-ratios and/or input-ratios.

The rest of this paper is organized as follows: Section 2 shows how the conventional DEA models with ratio data may give incorrect scores. Section 3 introduces some new DEA models to rectify the problem in the presence of output-ratio. Section 4 gives similar models for the case of input-ratio. This is followed by Section 5 to introduce a model for DEA with input-ratio and output-ratio. The conclusion remarks as well as points for future research are given in the last section.

2. Problem with ratio as output (output-ratio)

DEA is designed for evaluating DMUs that perform similar tasks and for which measurement of inputs and outputs are available [16]. Consider the case where each DMU consumes m inputs to produce$s$ outputs. Assume a set of n observed DMUs, DMU_j (for each $j = 1, \ldots, n$) is associated with input vector of $\boldsymbol{x}_j = (x_{1j}, \ldots, x_{mj})$ and output vector of $\boldsymbol{y}_j = (y_{1j}, \ldots, y_{sj})$. Also, let P be the production possibility set, i.e.

$$P = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \textbf{ can produce } \mathbf{y}\}.$$

In many applications, the production possibility set is unknown. The DEA approaches therefore estimate P from the set of observed DMUs and evaluate the observed productions relative to the estimated technology. Convexity is one of the underlying assumptions for estimating P. The following example shows how the convexity assumption may fail when at least one of the input or output variables is ratio.

Example 1. A case of mix output, absolute and output-ratio: For the purpose of this example consider a case where three universities have used a single input (total expenditure) of 1 to produce two outputs, $Y_1 = \%$ degree awarded and $Y_2 =$ amounts of research income (in million £).

Further, assume that % degree awarded (which is in the form of output-ratio) has been obtained from the following background information.

According to convexity axiom of DEA, the convex combination of U_2 and U_3 should also be a feasible university (i.e. belong to the production possibility set). The production possibility set (in this case; output set) is illustrated in Fig. 1.

The convex combination of U_2 and U_3 is $\lambda y_2 + (1 - \lambda) y_3$. Assume $\lambda = 0.5$ and denote their corresponding convex combination as U_{23} which should also be a feasible university, hence in the PPS. However the actual convex combination of two universities indicates that % degree awarded for university U_{23} should be calculated as follows:

$$\begin{aligned}
\%\text{ degree awarded for } U_{23} &= \frac{\text{Number of degree awarded for } U_{23}}{\text{Number of students for } U_{23}} \\
&= \frac{\frac{1}{2}(\text{Number of degree awarded for } U_2 + \text{Number of degree awarded for } U_3)}{\frac{1}{2}(\text{Number of students for } U_2 + \text{Number of students for } U_3)} \\
&= \frac{770}{7000} = 11\%.
\end{aligned}$$