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Solvability of the 3D rotating Navier–Stokes equations coupled with a 2D biharmonic problem with obstacles and gradient restriction ☆

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ABSTRACT

A coupled system by the 3D rotating Navier–Stokes equations with a mixed boundary condition and a 2D biharmonic problem with two obstacles and the gradient restriction is investigated in this paper. Using the Schauder's fixed point theorem, we show the existence of a strong solution for a sufficiently large viscosity ν and sufficiently small data.

1. Introduction

Recently, the coupled systems for fluids and solids have been investigated from theoretical and numerical points of view by many scholars. Such systems have an important application in engineering. A general reference can be found in [1,2]. Some different problems according to different physical models arise, such as elementary fluids in [3], ideal fluids in [4–6], viscous incompressible fluids in [7–15], and references therein.

In [12], Quarteroni, Tuver and Veneziani established a mathematical model describing the blood flow in arteries, which was a system coupled by the 2D Navier–Stokes equations and a 1D fourth-order elastic evolution problem. In their paper, an appropriate numerical method, the arbitrary Lagrangian–Eulerian method, was developed. Concerning the well-posedness of this system, Veiga in [7] proved the existence of a local strong solution in a small time interval under some assumptions on small initial data. The method used there is the Schauder's fixed point theorem after changing the domain into a fixed domain by the change of variables. But in these papers, the model of fluids is the 2D Navier–Stokes equations. Chambolle, Desjardins, Esteban and Grandmont in [13] considered a system coupled by the 3D Navier–Stokes equations and a 2D elastic evolution problem. They showed the existence of a weak solution without any assumptions on the small data. Moreover, the weak solution is global if the domain is always connected at any time. The main tool in there is the Schauder's fixed point theorem after replacing the change of variables by a linearized and regularized method. On a steady-state fluid–structure problem, Grandmont in [16] dealt with the coupling of the 2D Stokes equations, a second-order problem, and a fourth-order problem, and showed the existence of a weak solution by the change of variables and the Schauder's fixed point theorem.

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The model in this paper is from the mathematical model of the optimal shape design for blade surfaces of an impeller in [17,18], where the authors show that the mathematical model of the optimal shape design is a coupled system by the 3D rotating Navier–Stokes equations with a mixed boundary condition and a 2D fourth-order nonlinear elliptic problem. In this paper, we simplify the mathematical model in [17,18] and consider a coupled system by the 3D rotating Navier–Stokes equations with a mixed boundary condition and a 2D biharmonic problem with two obstacles.

It is well known that when the Navier–Stokes equations for a steady viscous incompressible fluid are written in a rotating frame of reference, two new terms appear (e.g. [19]). One of them is the centrifugal force which can be written as $\vec{\omega} \times (\vec{\omega} \times r)$, where $\vec{\omega}$ is the angle velocity of rotation of the frame of reference and r is the vector of position of the particles referred to this system. This term can be included in the external force. The other term is the Coriolis force which can be expressed as $2\vec{\omega} \times u$. Hence, in this case, the steady rotating Navier–Stokes equations can be written as

$$\begin{cases} -\nu \Delta u + (u \cdot \nabla)u + 2\vec{\omega} \times u + \nabla p = f_0 & \text{in} \quad \Omega, \\ \operatorname{div} u = 0 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where u denotes the velocity of fluid, p is a scalar function and denotes the pressure, f_0 includes the external force and the centrifugal force, $\Omega \subset \mathbb{R}^3$ is a bounded connected domain with boundary $\partial\Omega$ composed by fourth components Γ_0, Γ_N, Γ_t and Γ_b, here Γ_N is the inflow and outflow boundary, Γ_0 is fixed and denotes solid walls, Γ_t and Γ_b are the top blade surface and bottom blade surface, respectively, which are determined by unknown function $\Phi(x,y)$ and $\Theta(x,y)$. In terms of the change of variables, Γ_t and Γ_b can be separately dealt with by similar methods. Hence, for simplicity, we will assume that Γ_t is only unknown. After this assumption, the domain Ω can be simply represented as

$$\Omega = \{(x,y,z) \mid (x,y) \in D, 0 < z < \Phi(x,y)\},$$

where $D \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. The mixed boundary conditions associated with (1) are

$$\begin{cases} u = 0 & \text{on } \Gamma_0, \\ u(x,y,0) = 0 & \text{on } \Gamma_b, \\ u(x,y,\Phi(x,y)) = 0 & \text{on } \Gamma_t, \\ -\nu \frac{\partial u}{\partial n} + pn = g_0 & \text{on } \Gamma_N, \end{cases} \tag{2}$$

where n denotes the unit external normal to Γ_N.

Another problem is satisfied by the shape function $\Phi(x,y)$ of the blade. For given $g_1 \in H^2(D)$, let

$$\mathbb{K} = \left\{ w \in H^2(D), \quad w - g_1 \in H_0^2(D), \quad \psi_1(x,y) \leqslant w(x,y) \leqslant \psi_2(x,y), \quad |\nabla w| \leqslant \frac{1}{2} \text{a.e. in } D \right\},$$

where $0 < \psi_i(x,y) \in H^4(D)$. Let the shape function $\Phi(x,y)$ minimize the following energy functional

$$J(\phi) = \frac{1}{2} \int_D [|\Delta\phi(x,y)|^2 + \phi^2(x,y) + f_1(x,y,U) \cdot \phi(x,y)] \mathrm{d}x\mathrm{d}y, \tag{3}$$

on $\mathbb{K}$, where $f_1(x,y,U)$ denotes the external force and is a nonlinear function with respect to $U = \int_0^\Phi u(x,y,z)\mathrm{d}z$, u is the solution of (1), and (2). The physical meaning of the functional J is the mechanical energy of the moving fluid boundary. Then $\Phi(x,y)$ satisfies the biharmonic obstacle problem. The reason that we consider the biharmonic obstacle problem is that we require the shape function $\Phi(x,y)$ lying the space occupied by two obstacles $\psi_1(x,y)$ and $\psi_2(x,y)$. Moreover, the slope of the shape function $\Phi(x,y)$ is bounded by a given value. It is obvious that (1)–(3) is a coupled system by the shape function $\Phi(x,y)$. In this paper, we will deal with the existence of a strong solution of the coupled system (1)–(3).

To ensure that Ω is connected, we require

$$0 < \psi_1(x,y) \leqslant \Phi(x,y) \leqslant \psi_2(x,y) \quad \forall (x,y) \in D. \tag{4}$$

Since $\Phi(x,y)$ is unknown, the domain Ω will be changed into a fixed domain by the change of variables. From some references, such as [7,16], we know that the fixed point method is a powerful tool for the coupled system, which will be applied to our problem. Unlike unsteady problem, for which the small time T plays a key role, the difficulty of our problem lies in selecting an appropriate closed convex set.

This paper is organized as follows: in Section 2, we give some preliminary notation and state the main result; in Section 3, the change of variables is applied; in Section 4, we will show existence of a weak solution of the rotating Navier–Stokes problem in the new coordinates; in Section 5, the regularity of the solution of the obstacle is obtained; finally, in Section 6, the fixed point procedure is described.

2. Preliminary notation and main result

Denote by $W^{m,p}(\Omega)$ the standard Sobolev space with norm $\|u\|_{m,p}$. In particular, if $p = 2$, we write $H^m(\Omega) = W^{m,p}(\Omega)$. Moreover, for simplicity, set $\|u\|_m = \|u\|_{m,2}$. The inner product and norm in $L^2(\Omega)$ are equipped by $(\cdot,\cdot)$ and $\|\cdot\|_0$, respectively. Define

$$E = \{u \in C^\infty(\Omega)^3, \quad \operatorname{supp} u \cap (\Gamma_0 \cup \Gamma_t \cup \Gamma_b) = \emptyset\}.$$