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Analysis for an *N*-stepped Rayleigh bar with sections of complex geometry

Igor Fedotov [a], Ying Gai [a,*], Andrei Polyanin [b], Michael Shatalov [a,c]

[a] *Department of Mathematical Technology, Tshwane University of Technology, Private Bag X680, Pretoria 0001, South Africa*
[b] *Institute for Problems in Mechanics of Russian Academy of Science, 101, Vernadskogo Prospect, Moscow, Russia*
[c] *Sensor Science and Technology (SST) of CSIR Material Science and Manufacturing, P.O. Box 395, Pretoria 0001, CSIR, South Africa*

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Abstract

In this paper we analyse the vibrations of an *N*-stepped Rayleigh bar with sections of complex geometry, supported by end lumped masses and springs. Equations of motion and boundary conditions are derived from the Hamilton's variational principal. The solutions for tapered and exponential sections are given. Two types of orthogonality for the eigenfunctions are obtained. The analytic solution to the *N*-stepped Rayleigh model is constructed in terms of Green function.
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1. Introduction

To model the mechanical vibrations in acoustical transducers, wave equation are often used for description of the effects of wave propagation, supposing that the sections have lengths much longer than the characteristic dimension of a cross-section. In this approach, the lateral effects are neglected in comparison to the major longitudinal vibrations. It was shown by Rayleigh (see [1]) that the error caused by ignoring the lateral motion, is proportional to the square ratio of the characteristic radius of a cross-section to the bar's length. For proper investigation of longitudinal vibrations of an *N*-stepped bar, it is necessary to take into account the lateral effects. In the present paper one-dimensional longitudinal vibrations of an *N*-stepped bar with non-uniform cross-sections, supported by end lumped masses and springs, is considered in the frames of Rayleigh theory, where the effects of lateral inertial are considered.

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[*] Corresponding author.
E-mail addresses: fedotovi@tut.ac.za (I. Fedotov), yingandy2006@yahoo.com (Y. Gai), polyanin@ipmnet.ru (A. Polyanin), mshatlov@csir.co.za (M. Shatalov).

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Equations of motion and boundary conditions for the vibrations of Rayleigh bar with cross-sections of complex geometry, are derived from the Lagrangian and the Hamilton's variational principle (cf [2]). The eigenvalues are found from the characteristic system, and two sorts of orthogonality exist for the eigenfunctions. The exact solutions for sections with particular geometry are found. For example, in the case of a tapered section, the solution is obtained in terms of the Legendre functions of the first and second kind. In the case of an exponential section the solution is obtained in terms of the Gauss hypergeometric functions. Hence, we obtained two absolutely new cases of exact integration of equations of motion of the mechanical system which are described in terms of partial differential equations. It is significant to note that the exact solution to the Rayleigh equation is of a great interest even in the case of a one section bar.

In the sequel we construct the analytical solution to the N-stepped Rayleigh problem in terms of a Green function. The proposed theoretical analytical approach differs from the FE simulation approach, the latter being an experimental method, namely the numerical experiment with discrete idealized models. Without doubt the problem could be solved by using the available FE software technology but we strongly believe that the best way to solve any problem of mathematical physics is to use an analytical method. In most cases by "analytical solution" we mean a solution expressed in terms of the Green function. The major advantage of an analytical solution is its easy, transparent and accurate analysis of the physical nature of a problem. If a purely analytical method is not available we can also try to use a combination of analytical approximations with numerical methods.

In greater detail, the main idea of using the analytical approach and our motivations are as follows:

1. Existing one-dimensional finite elements for analysis of longitudinal vibrations of solid bars are based on the model of wave equation and do not take into consideration the effects of lateral inertia.
2. Existing two-dimensional axisymmetric finite elements as well as three-dimensional elements give redundant information on vibration of multi-stepped structure, which is very important on the stage of practical realization of transducer design. However it is not always necessary on a preliminary stage of the solid state waveguides and transducers design, such as underwater acoustic transducers of Tonpilz type, power ultrasound transducers for sonochemistry and medical applications, etc.
3. The proposed analytical approach constitutes a substantial progress in the designing methods in comparison with the traditional techniques (see, for example, [3]).
4. Available commercial FE software is not properly suited for the purposes of transducers optimization. It does not give versatile tools to designer for fast and easy changing of geometrical and physical parameters of the model and takes substantial time for the "preprocessing – calculation – post-processing" cycle (minutes instead of seconds taken by the proposed analytical approach!). Last but not least this software is very expensive for small companies dealing with design and manufacturing of transducers in the countries of the third world, for example, in Africa. The proposed analytical approach is most appropriate for the purposes of the transducers optimization and available as a simple and fast interactive exe-file for design of powerful transducers.

Finally, an example of forced vibration of a three-stepped Rayleigh bar is given. Seven eigenvalues are determined and corresponding non-smooth eigenfunctions are drawn. It is shown that the convergence is very fast because the reaction of the bar is practically the same as in the cases of three, five and seven eigenfunctions.

2. Equation with boundary conditions, eigenvalues and eigenfunctions, orthogonality of eigenfunctions

2.1. Equation of motion of an N-stepped Rayleigh bar with sections of complex geometry

Let us consider the longitudinal vibrations of an N-stepped bar with sections of complex geometry (Fig. 1):

Suppose that the length of each section of the bar is comparable with the linear dimension of its cross-section, and the left and right ends of each section is attached to a rigid medium by lumped masses and springs (K_j, M_j). The N-stepped bar is excited by a distributed force $F(x,t)$. We denote the longitudinal displacements of the stepped bar by $u = u(x,t)$. In the present paper, the effects of lateral inertias are taken into consider-