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A linearized compact difference scheme for an one-dimensional parabolic inverse problem

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Abstract

The parabolic equation with the control parameter is a class of parabolic inverse problems and is nonlinear. While determining the solution of the problems, we shall determinate some unknown control parameter. These problems play a very important role in many branches of science and engineering. The article is devoted to the following parabolic initial-boundary value problem with the control parameter: $\partial u/\partial t = \partial^2 u/\partial x^2 + p(t)u + \phi(x,t), 0 < x < 1, 0 < t \leqslant T$ satisfying $u(x,0) = f(x), 0 < x < 1; u(0,t) = g_0(t), u(1,t) = g_1(t), u(x^*,t) = E(t), 0 \leqslant t \leqslant T$ where $\phi(x,t), f(x), g_0(t), g_1(t)$ and $E(t)$ are known functions, $u(x,t)$ and $p(t)$ are unknown functions. A linearized compact difference scheme is constructed. The discretization accuracy of the difference scheme is two order in time and four order in space. The solvability of the difference scheme is proved. Some numerical results and comparisons with the difference scheme given by Dehghan are presented. The numerical results show that the linearized difference scheme of this article improve the accuracy of the space and time direction and shorten computation time largely. The method in this article is also applicable to the two-dimensional inverse problem.

Keywords: Control parameter; Parabolic equation; Difference scheme; Compact difference scheme; Solvability

1. Introduction

In this article, we study the numerical solution to the following inverse problem. Find $u(x,t)$ and $p(t)$ which satisfy

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + p(t)u + \phi(x,t), \quad 0 < x < 1,\ 0 < t \leqslant T, \tag{1.1}$$

with initial condition

$$u(x,0) = f(x), \quad 0 < x < 1, \tag{1.2}$$

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and boundary conditions

$$u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad 0 \leqslant t \leqslant T \tag{1.3}$$

subject to the over-specification at a point in the spatial domain

$$u(x^*,t) = E(t), \quad 0 \leqslant t \leqslant T \tag{1.4}$$

where $f(x),\ g_0(t),\ g_1(t),\ \phi(x,t)$ and $E(t)$ are known functions, $|E(t)| \geqslant E_0 > 0,\ x^* \in (0,1)$, while $u(x,t)$ and $p(t)$ are unknown functions.

If $u(x,t)$ represents temperature then the problem (1.1)–(1.4) can be viewed as a control problem of finding the control $p(t)$ such that the internal constraint (1.4) is satisfied.

This kind of inverse problems of parabolic type arises from various fields of science and engineering. The existence and uniqueness of the solutions to these problems and also some more applications are discussed in [1–10].

Cannon et al. [11] formulated a backward Euler finite difference scheme via a transformation and proved the convergence of u with the convergence order of $O(\tau + h^2)$ and of p with the convergence order of $O(\tau^{1/2})$ when $\tau = O(h^2)$. Dehghan in [12] presented for the problem (1.1)–(1.4) four difference schemes. They are three-point explicit scheme, five-point explicit scheme, three-point implicit scheme and implicit Crandall's scheme. But there is no theoretical justification. In [13,14], Dehghan presented the numerical methods for (1.1)–(1.3) with the integral over-specification of the function $k(x)u(x,t)$ over the spatial domain

$$\int_0^1 k(x)u(x,t)\mathrm{d}x = E(t), \quad 0 \leqslant t \leqslant T,$$

or

$$\int_0^{s(t)} k(x)u(x,t)\mathrm{d}x = E(t), \quad 0 \leqslant t \leqslant T,\ 0 < s(t) < 1,$$

respectively. Dehghan [15–17] and Daoud and Subasi [18] presented some numerical methods for solving two-dimensional inverse control problem.

Take two positive integers M and N. Let $h = 1/M,\ \tau = T/N$ and $r = \tau/h^2$. Denote $\Omega_h = \{x_i \mid x_i = ih,\ 0 \leqslant i \leqslant M\}$, $\Omega_\tau = \{t_n \mid t_n = n\tau,\ 0 \leqslant n \leqslant N\}$, $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. Suppose that there is an integer k_0 such that $x^* = x_{k_0}$. In practical problems, this is always possible as pointed out in [12].

For a mesh function $\{u_i^n | 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N\}$ on $\Omega_{h\tau}$, introduce the following notations:

$$u_i^{n+\frac{1}{2}} = \frac{u_i^{n+1} + u_i^n}{2}, \quad \delta_t u_i^{n+\frac{1}{2}} = \frac{u_i^{n+1} - u_i^n}{\tau}, \quad D_{\hat{t}} u_i^n = \frac{u_i^{n+1} - u_i^{n-1}}{2\tau},$$
$$u_i^{\bar{n}} = \frac{u_i^{n+1} + u_i^{n-1}}{2}, \quad \delta_x^2 u_i^n = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2},$$
$$\|u^n\| = \left[h\sum_{i=1}^{M-1}(u_i^n)^2\right]^{1/2}, \quad \|u^n\|_\infty = \max_{0 \leqslant i \leqslant M} |u_i^n|.$$

Define the mesh functions

$$\Phi_i^n = \phi(x_i,t_n), \qquad E^n = E(t_n), \qquad (E')^n = E'(t_n), \quad 0 \leqslant i \leqslant M, \quad 0 \leqslant n \leqslant N.$$

Dehghan proposed the following implicit Crandall's scheme for problem (1.1)–(1.4) in [12]:

$$\frac{1}{12}\left(\delta_t u_{i-1}^{n+\frac{1}{2}} + 10\delta_t u_i^{n+\frac{1}{2}} + \delta_t u_{i+1}^{n+\frac{1}{2}}\right) = \delta_x^2 u_i^{n+\frac{1}{2}} + p^{n+1}u_i^n + \Phi_i^{n+1}, \quad 1 \leqslant i \leqslant M-1,\ 0 \leqslant n \leqslant N-1, \tag{1.5}$$

$$p^{n+1} = \frac{1}{E^{n+1}}\left[(E')^{n+1} - \frac{1}{12h^2}\left(-u_{k_0-2}^{n+1} + 16u_{k_0-1}^{n+1} - 30u_{k_0}^{n+1} + 16u_{k_0+1}^{n+1} - u_{k_0+2}^{n+1}\right) - \Phi_{k_0}^{n+1}\right], \quad 0 \leqslant n \leqslant N-1, \tag{1.6}$$