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On the additive splitting procedures and their computer realization

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Abstract

Two additive splitting procedures are defined and studied in this paper. It is shown that these splitting procedures have good stability properties. Some other splitting procedures, which are traditionally used in mathematical models used in many scientific and engineering fields, are sketched. All splitting procedures are tested by using six different numerical methods for solving differential equations. Many conclusions, which are related both to the comparison of the additive splitting procedures with the other splitting procedures and to the influence of the numerical methods for solving differential equations on the accuracy of the splitting procedures, are drawn.

Keywords: Operator splitting; Numerical methods; Time discretization; Stability

1. Statement of the problem

In our investigation we will assume that there are only two operators, i.e. we will demonstrate our methods on the (abstract) Cauchy problem of the form:

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = (A+B)w(t), \quad t \in (0,T], \ \ w(0) = w_0. \tag{1}$$

The theoretical results are proved under an assumption that the involved operators are bounded linear operators, and hence the exact solution is $w(t) = \exp(t(A+B))w(0)$. The experimental results indicate that good results should also be expected when this assumption is relaxed.

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2. Traditional operator splitting methods

In this section we give a short overview of the different known operator splitting methods which are used in many different applications. For more details, see [1–3].

2.1. Sequential splitting

If the sub-problems involving operators A and B from (1) are treated one after the other, then the resulting algorithm is called a sequential splitting procedure.

It is very often worthwhile to describe how the different splitting procedures are to be applied in practice by

- giving the order in which the simple operators A and B are applied, and
- indicating the splitting time-step size τ, which is actually used.

In our particular case, see again (1), the application of the sequential splitting procedure at a given splitting time-step can be described by sequence

$$(A)_\tau, \quad (B)_\tau. \tag{2}$$

It is necessary to explain how the two sub-models are coupled. Assume that n splitting time-steps have successfully been performed and the next splitting time-step, time-step $n+1$, has to be carried out. The approximation obtained at time-step n is used as a starting approximation when the first sub-model is treated. The approximation obtained at the end of computations related to the first sub-model is used as a starting value for the second sub-model. The approximation obtained when the computations related to the second sub-model are accomplished is accepted as an approximation of the solution of problem (1) at time-step $n+1$. In this way everything is prepared to start the computations related to time-step $n+2$. It is necessary to explain how to start the computations at time-step 1, but this is not causing problems, because it is assumed that $w(0)=w_0$ is given; see again (1).

It should be noted here that if we change the order of the application of the operators, then the results will normally not be the same, i.e. the sequence $(A)_\tau$, $(B)_\tau$ is in general different from the sequence $(B)_\tau$, $(A)_\tau$.

The sequential splitting is in general leading to a numerical approximation of order one. The implication of this fact is that as a rule it is not advisable to use numerical algorithms of order higher than one in the treatment of the sub-problems involving the simpler operators A and B when a sequential splitting procedure is to be used.

2.2. Marchuk–Strang splitting

Sometimes it is desirable to apply more accurate splitting procedures. Accuracy of order two can be achieved in the following way. Consider an arbitrary splitting time-step, say step n (i.e. the computations are to be carried out from $t=t_n$ to $t=t_{n+1}=t_n+\tau$). Assume that the sub-models are treated as follows:

- Carry out computations by using the first operator from $t=t_n$ to $t=t_n+0.5\tau$.
- Use the second operator to perform computations from $t=t_n$ to $t=t_n+\tau$.
- Perform computations from $t=t_n+0.5\tau$ to $t=t_n+\tau$ by applying again the first operator.

This splitting procedure was proposed in 1968 simultaneously by Marchuk and Strang (see [4,5]). It is also called symmetric splitting.

In the notation used in the previous sub-section the calculations at an arbitrary splitting time-step can be described by the sequence:

$$(A)_{0.5\tau}, \quad (B)_\tau, \quad (A)_{0.5\tau}. \tag{3}$$