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On the sign-stability of numerical solutions of one-dimensional parabolic problems ☆

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Abstract

The preservation of the qualitative properties of physical phenomena in numerical models of these phenomena is an important requirement in scientific computations. In this paper, the numerical solutions of a one-dimensional linear parabolic problem are analysed. The problem can be considered as a altitudinal part of a split air pollution transport model or a heat conduction equation with a linear source term. The paper is focussed on the so-called sign-stability property, which reflects the fact that the number of the spatial sign changes of the solution does not grow in time. We give sufficient conditions that guarantee the sign-stability both for the finite difference and the finite element methods.

Keywords: Parabolic problems; Numerical solution; Qualitative properties; Sign-stability

1. Introduction

The concentration of r air-pollutants can be modelled and forecasted by the so-called air pollution transport model (e.g., [1]), which has the form

$$\frac{\partial v_l}{\partial t} = -\nabla(\mathbf{u}v_l) + \nabla(k\nabla v_l) + R_l + E + \sigma v_l \quad (l = 1, \ldots, r). \tag{1}$$

Here the unknown function $v_l = v_l(x, y, z, t)$ is the concentration of the lth pollutant, the function $\mathbf{u} = \mathbf{u}(x, y, z, t)$ describes the wind velocity, $k = k(x, y, z, t)$ is the diffusion coefficient, $R_l = R_l(x, y, z, t, v_1, \ldots, v_l)$ describes the chemical reactions between the investigated pollutants, $E = E(x, y, z, t)$ is the emission function and $\sigma = \sigma(x, y, z, t)$ describes the deposition process.

Because of its complexity, system (1) is generally solved applying the so-called operator splitting technique. The system is split into several subproblems according to the physical and chemical processes involved in the model: advection, diffusion, chemical reaction, emission and deposition. These subproblems are solved

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cyclically with some appropriate methods (e.g., [2,3]). Then, the solution of the model can be obtained using the solutions of the subproblems. Splitting, however, can be applied not only according to the physical or chemical processes but according to the space coordinates, too. In this case, for instance, the longitudinal and latitudinal space coordinates are handled separately from the altitudinal coordinate.

Naturally, the properties of the solution of an air pollution transport model are determined by the properties of the numerical methods that are applied for the subproblems. In order to get a qualitatively correct numerical solution of the whole model, the subproblems must be solved with qualitatively adequate numerical methods, too.

In this paper, we consider the one-dimensional problem

$$\frac{\partial v}{\partial t} - \frac{\partial}{\partial x}\left(\kappa \frac{\partial v}{\partial x}\right) + \gamma v = 0, \quad (x,t) \in (0,1) \times (0,\infty), \tag{2}$$

$$v(x,0) = v_0(x), \quad x \in (0,1), \tag{3}$$

$$v(0,t) = v(1,t) = 0, \quad t \geqslant 0, \tag{4}$$

where v_0 is a given continuous initial function, the continuous function $\gamma : [0,1] \to \mathbb{R}$ possesses the property

$$0 < \gamma_{\min} \leqslant \gamma(x) \leqslant \gamma_{\max};$$

the function $\kappa : [0,1] \to \mathbb{R}$ fulfills the property

$$0 < \kappa_{\min} \leqslant \kappa(x) \leqslant \kappa_{\max}$$

and it has continuous first derivatives. A function $v : [0,1] \times \mathbb{R}_0^+ \to \mathbb{R}$ is called the solution of problem (2)–(4) if it is sufficiently smooth and satisfies equalities (2)–(4). Eq. (2) can be considered as one of the subproblems of an air pollution transport model, namely the one that describes the altitudinal changes. Eq. (2) is also suitable to describe heat conduction processes. In this case v denotes the temperature, κ is the heat conduction coefficient and $-\gamma v$ is a linear source term.

The investigation of the number of the sign changes of real functions goes back as far as to 1836 [4]. Then in the thirties, Pólya [5] showed for the case $\gamma = 0$, $\kappa = 1$ that the number of the sign-changes of the function $x \mapsto v(x,t)$ ($x \in [0,1]$) does not grow in t. This property is called sign-stability.

A number of qualitative properties of Eq. (2), such as nonnegativity preservation, maximum–minimum principle, maximum norm contractivity, etc, are thoroughly investigated in the literature both for finite difference and finite element methods [6–10]. This is not the case for the sign-stability property. There are sufficient conditions given for the finite difference solution of (2), but for the finite element method there are no results available.

It was shown in [11] that if the relation

$$\frac{\tau}{h^2} \leqslant \frac{1}{4(1-\theta)} \tag{5}$$

is satisfied then the finite difference method with the uniform spatial step-size h and with the θ time-discretization method with the time-step τ is sign-stable for problems where $\gamma = 0$ and $\kappa = 1$. If $\theta = 1$, then there is no upper bound for the quotient τ/h^2. In [12], we showed that condition (5) is the necessary and sufficient condition of the uniform (independent of h) sign-stability, and we extended the investigation to finite element solutions. In [13], a sufficient condition of the sign-stability is given for the explicit finite difference solution of a semi-linear parabolic problem. The proof is based on a six-page-long linear algebraic consideration about the sign-stability of positive tridiagonal matrices. Paper [9] simplifies the proof essentially and gives sufficient conditions of the sign-stability of the finite difference methods applied for (2).

2. Sign-stability of tridiagonal matrices

Let n be a fixed natural number. In order to simplify the notations, we introduce the sets $N = \{1,\ldots,n\}$, $J_i = \{i-1, i, i+1\}$ $(i = 2,\ldots,n-1)$, $J_1 = \{1,2\}$ and $J_n = \{n-1,n\}$. The elements of a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ are