Applied Mathematics Letters 64 (2017) 34–41

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Generalized Prüfer angle and oscillation of half-linear differential equations

O. Došlý[a], J. Jaroš[b], M. Veselý[a,*]

[a] Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic
[b] Department of Mathematical Analysis and Numerical Mathematics, Comenius University, 842 48 Bratislava, Slovakia

ARTICLE INFO

Article history:
Received 9 August 2016
Accepted 9 August 2016
Available online 22 August 2016

Keywords:
Prüfer angle
Half-linear equations
Oscillation theory
Conditional oscillation
Oscillation constant

ABSTRACT

In this paper, we introduce a new modification of the half-linear Prüfer angle. Applying this modification, we investigate the conditional oscillation of the half-linear second order differential equation

$$\left[t^{\alpha-1} r(t)\,\Phi(x')\right]' + t^{\alpha-1-p} s(t)\,\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \tag{$*$}$$

where $p > 1$, $\alpha \neq p$, and r, s are continuous functions such that $r(t) > 0$ for large t. We present conditions on the functions r, s which guarantee that Eq. $(*)$ behaves like the Euler type equation $[t^{\alpha-1}\Phi(x')]' + \lambda t^{\alpha-1-p}\Phi(x) = 0$, which is conditionally oscillatory with the oscillation constant $\lambda_0 = |p-\alpha|^p/p^p$.

1. Introduction

The half-linear differential equation is an equation of the form

$$[R(t)\,\Phi(x')]' + S(t)\,\Phi(x) = 0 \tag{1.1}$$

with the odd power function $\Phi(x) = |x|^{p-1}\operatorname{sgn} x$ for some $p > 1$ and with continuous coefficients R, S, where R is positive. The oscillation theory of Eq. (1.1) has attracted considerable attention in the recent years. We refer to papers below and books [1,2] for references up to the publication years of these books. One of the reasons for this interest is that the qualitative theory of Eq. (1.1) is very similar to that of the Sturm–Liouville differential equation $[R(t)x']' + S(t)x = 0$. This linear equation is the particular case of Eq. (1.1) for $p = 2$ and its qualitative theory is very deeply developed. In particular, the classical Sturmian oscillation theory for linear equations is extended to Eq. (1.1). This means that Eq. (1.1) can be classified as oscillatory or non-oscillatory (analogously as in the linear case).

* Corresponding author.
E-mail address: xvesely@math.muni.cz (M. Veselý).

http://dx.doi.org/10.1016/j.aml.2016.08.004

One of the half-linear differential equations which can be solved explicitly (at least partially) is the Euler half-linear differential equation

$$\left[t^{\alpha-1}\Phi(x')\right]' + \gamma t^{\alpha-p-1}\Phi(x) = 0, \quad \alpha \neq p. \tag{1.2}$$

Especially, it is known (see [3,4] or Theorem A) that Eq. (1.2) is oscillatory if $\gamma p^p > |p-\alpha|^p$, and non-oscillatory in the opposite case. Eq. (1.2) is a typical example of the so-called *conditionally oscillatory* half-linear differential equation. Let us consider Eq. (1.1), where $S(t) = \lambda c(t)$ for $\lambda \in \mathbb{R}$ and a continuous function c. We recall that Eq. (1.1) is said to be conditionally oscillatory if there exists the so-called *oscillation constant* $\lambda_0 > 0$ such that Eq. (1.1) with $S(t) = \lambda c(t)$ is oscillatory for $\lambda > \lambda_0$ and non-oscillatory for $\lambda < \lambda_0$.

Concerning the conditional oscillation, a natural question is what happens when one considers the more general equation in the form

$$\left[t^{\alpha-1}r(t)\Phi(x')\right]' + t^{\alpha-p-1}s(t)\Phi(x) = 0. \tag{1.3}$$

We remark that Eq. (1.2) is the special case of Eq. (1.3) with $r(t) \equiv 1$, $s(t) \equiv \gamma$. This problem is partially studied, e.g., in [5–10] (and, e.g., in [11–16] for the linear case). In this paper, we introduce a new very general modification of the half-linear Prüfer angle. Using this concept of the Prüfer angle, we obtain conditions on the functions r, s which guarantee that Eq. (1.3) remains conditionally oscillatory; i.e., we show that Eq. (1.3) behaves essentially in the same way as Eq. (1.2).

This paper is organized as follows. In the next section, we present the modified Prüfer transformation which is the main tool in our paper. Section 3 presents our main result about the conditional oscillation of Eq. (1.3).

2. Generalized Prüfer angle

Throughout this paper, we will consider all equations for large t, say in an interval $[b, \infty)$. Intervals of this form will be denoted by $\mathbb{R}_b$. Let $p > 1$ and $\alpha \in \mathbb{R} \setminus \{p\}$ be arbitrarily given. The symbol q will denote the number conjugated with p which means that q is given by the equality $p + q = pq$. We put

$$\gamma_{p,\alpha} := \left(\frac{|p-\alpha|}{p}\right)^p. \tag{2.1}$$

Let us consider the half-linear differential equation of the form

$$[R(t)\Phi(x')]' + S(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \tag{2.2}$$

where $R, S : \mathbb{R}_b \to \mathbb{R}$ are continuous functions and R is positive. Let x be a non-trivial solution of (2.2). For $x(t) \neq 0$, the so-called Riccati transformation $w(t) = R(t)\Phi(x'(t)/x(t))$ leads to the Riccati equation

$$w'(t) + S(t) + (p-1)R^{1-q}(t)|w(t)|^q = 0. \tag{2.3}$$

Concerning the correctness of the used transformations, we can refer to [2, Sections 1.1.3–1.1.6].

Let an arbitrary positive function $f \in C^1(\mathbb{R}_b)$ be given. The substitution $v(t) = f^p(t)w(t)$ and Eq. (2.3) give

$$v'(t) = pf^{p-1}(t)f'(t)w(t) + f^p(t)w'(t) = p\frac{f'(t)}{f(t)}v(t) - f^p(t)S(t) - (p-1)R^{1-q}(t)f^{-q}(t)|v(t)|^q. \tag{2.4}$$

Using the above function f, we introduce the modified Prüfer transformation

$$x(t) = \rho(t)\sin_p\varphi(t), \quad R^{q-1}(t)x'(t) = \frac{\rho(t)}{f^q(t)}\cos_p\varphi(t) \tag{2.5}$$