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On the reduction of multivariate quadratic systems to best rank-1 approximation of three-way tensors

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ABSTRACT

In this paper, we show that a general quadratic multivariate system in the real field can be reduced to a best rank-1 three-way tensor approximation problem. This fact provides a new approach to tackle a system of quadratic polynomials equations. Some experiments using the standard alternating least squares (ALS) algorithm are drawn to evince the usefulness of rank-1 tensor approximation methods.

1. Introduction

Let P1 be the general system of quadratic polynomial equations given by

$$\mathbf{P1}: \begin{cases} \boldsymbol{x}^\mathsf{T}\boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{b}_1^\mathsf{T}\boldsymbol{x} + c_1 = 0 \\ \vdots \\ \boldsymbol{x}^\mathsf{T}\boldsymbol{A}_m\boldsymbol{x} + \boldsymbol{b}_m^\mathsf{T}\boldsymbol{x} + c_m = 0, \end{cases}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{A}_j \in \mathbb{R}^{n\times n}$ are symmetric matrices, $\boldsymbol{b}_j \in \mathbb{R}^n$ and $c_j \in \mathbb{R}$, $1 \le j \le m$.

To tackle this problem, some mathematical techniques can be applied, such as Newton and tensor-based algorithms [1,2], Gröbner bases, resultants and eigenvalues/eigenvectors of companion matrices [3], semidefinite relaxations [4–6], numerical homotopy [7,8], low-rank matrix recovery [9], and symbolic computation [10]. The complexity class of System (P1) is NP-hard.

This problem is of great interest in various applications. For instance, in game theory, the Nash equilibria of a non-cooperative game between two players can be found by solving a multivariate quadratic system [11].

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In cryptography, the security of systems depends on the difficulty to solve large quadratic systems in finite fields [12,13]. In [14] the authors present the design of multivariate filter banks modeled by quadratic polynomial systems. A last example of application lies in multilinear algebra [15], where the authors propose an efficient algorithm to decompose a symmetric tensor and show the equivalence between an existence condition of the decomposition and the solution of a system of quadratic equations.

Let $\boldsymbol{\mathcal{T}} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ be a three-way tensor with entries in the real field. The best rank-1 approximation of $\boldsymbol{\mathcal{T}}$ can be obtained by solving the following optimization problem

$$p^\star = \min_{\boldsymbol{a}_i, 1 \le i \le N} \|\boldsymbol{\mathcal{T}} - \boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \boldsymbol{a}_3\|, \tag{1}$$

where $\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \boldsymbol{a}_3$ is a rank-1 tensor with approximating factors $\boldsymbol{a_i} \in \mathbb{R}^{n_i}$, $1 \le i \le 3$, $\otimes$ is the tensor product, and $\|\cdot\|$ is the Frobenius norm. The solution of Problem (1) always exists in $\mathbb{R}$, since the set of rank-1 tensors is closed [16]. It is known that standard tensor approximation algorithms, such as alternating least squares (ALS) [17] generally deliver satisfactory solutions in practice in reasonable time for the best rank-1 approximation problem. Other methods to solve this problem can be found in [6,18].

The goal of this paper is to show that multivariate quadratic systems can be reduced to best rank-1 three-way tensor approximation problems, which provides a new approach with a large amount of tensor tools to deal with System (P1).

2. From quadratic systems to best rank-1 approximations

Let $\mathscr{I}$ be the ideal in $\mathbb{R}[\boldsymbol{x}]$ generated by the set of polynomials of (P1), that is, $\mathscr{I} = \langle h_1(\boldsymbol{x}), h_2(\boldsymbol{x}), \ldots, h_m(\boldsymbol{x})\rangle$, where $h_j(\boldsymbol{x}) = \boldsymbol{x}^\mathsf{T}\boldsymbol{A}_j\boldsymbol{x} + \boldsymbol{b}_j^\mathsf{T}\boldsymbol{x} + c_j, 1 \le j \le m$. Thus, the set of solutions of (P1) is defined by the affine variety $V(\mathscr{I}) := \{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) = 0,\ \forall f \in \mathscr{I}\}$. Also define the optimization problem (P2) as follows

$$\mathbf{P2} : p_2^\star = \min_{\|\boldsymbol{y}\|=1} p(\boldsymbol{y})$$

where $p(\boldsymbol{y}) = \sum_{j=1}^m (\boldsymbol{y}^\mathsf{T}\boldsymbol{Q}_j\boldsymbol{y})^2$, for $\boldsymbol{y} \in \mathbb{R}^{n+1}$ and

$$\boldsymbol{Q}_j = \begin{bmatrix} \boldsymbol{A}_j & \boldsymbol{b}_j/2 \\ \boldsymbol{b}_j^\mathsf{T}/2 & c_j \end{bmatrix} \in \mathbb{R}^{n+1 \times n+1}.$$

Define the set of global minimizers of (P2) as $\mathscr{S}_{P2} = \{\boldsymbol{y} \in \mathbb{R}^{n+1} \mid p(\boldsymbol{y}) = p_2^\star, \|\boldsymbol{y}\| = 1\}$, and the subset $\bar{\mathscr{S}}_{P2} \subseteq \mathscr{S}_{P2}$ given by $\bar{\mathscr{S}}_{P2} = \mathscr{S}_{P2} \cap (\mathbb{R}^n \times \mathbb{R} \setminus \{0\})$. In other words, $\bar{\mathscr{S}}_{P2}$ is the set of solutions of Problem (P2) such that $y_{n+1} \neq 0$. Let also $\mathscr{N} = \{\boldsymbol{z} \in \mathbb{R}^{n+1} \mid \boldsymbol{z} = \boldsymbol{y}/y_{n+1},\ \forall \boldsymbol{y} \in \bar{\mathscr{S}}_{P2}\}$. Proposition below connects Problems (P1) and (P2).

Proposition 1. *If $V(\mathscr{I}) \neq \emptyset$ then $V(\mathscr{I}) \times \{1\} = \mathscr{N}$.*

Proof. By setting $\boldsymbol{y} = [\boldsymbol{x}\ 1]^\mathsf{T}$, it turns out that

$$\boldsymbol{x}^\mathsf{T}\boldsymbol{A}_j\boldsymbol{x} + \boldsymbol{b}_j^\mathsf{T}\boldsymbol{x} + c_j = \boldsymbol{y}^\mathsf{T}\boldsymbol{Q}_j\boldsymbol{y},$$

$\forall j \in \{1, 2, \ldots, m\}$. This shows that the set of solutions of the following system, equivalent to (P1),

$$\begin{cases} \boldsymbol{y}^\mathsf{T}\boldsymbol{Q}_j\boldsymbol{y} = 0, & \text{for } 1 \le j \le m \\ y_{n+1} = 1 \end{cases} \tag{2}$$

is $V(\mathscr{I}) \times \{1\}$. Now, consider the optimization problem (P2). Since $p(\boldsymbol{y})$ is by construction the sum of squares of the quadratic polynomials $\boldsymbol{y}^\mathsf{T}\boldsymbol{Q}_j\boldsymbol{y}$, $1 \le j \le m$, and $V(\mathscr{I}) \neq \emptyset$, it follows that $p_2^\star = 0$.