Applied Mathematics Letters 62 (2016) 16–22

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

A new blow-up criterion for Gross–Pitaevskii equation

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ARTICLE INFO

Article history:
Received 28 April 2016
Accepted 8 June 2016
Available online 22 June 2016

Keywords:
Gross–Pitaevskii equation
Invariant set
Blow-up

ABSTRACT

This paper is devoted to the study of the Gross–Pitaevskii equation, which describes the attractive Bose–Einstein condensate under a magnetic trap. By establishing an invariant set and applying the Gagliardo–Nirenberg type inequality, we then get the existence of blow-up solutions with arbitrary large initial energy.

1. Introduction

In this paper, we consider the Cauchy problem of the Gross–Pitaevskii equation

$$\begin{cases} iu_t + \Delta u - |x|^2 u + |u|^2 u = 0, \\ u(0,x) = u_0, \end{cases} \tag{1.1}$$

where $i^2 = -1,\ u = u(t,x) : [0,T) \times \mathbb{R}^3 \to \mathbb{C},\ 0 < T \le \infty$ and Δ is the Laplacian operator in $\mathbb{R}^3$. Eq. (1.1) models the remarkable Bose–Einstein condensate with attractive inter-particle interactions under a magnetic trap (see [1–5]).

The Gross–Pitaevskii equation (1.1) is a nonlinear Schrödinger equation with a harmonic potential. The nonlinear Schrödinger equation without potential has been studied extensively, see e.g. [6–8] for the global existence and [6,9–13] for the blow-up solutions.

Let us briefly comment the known results for the Cauchy problem (1.1). First of all, Oh [14] established the local well-posedness in corresponding energy space (also see [15]). Then, in view of [16,17], it is known that the solution of the Eq. (1.1) blows up in finite time if the initial energy is negative, i.e., $E[u_0] < 0$. Carles [18] proved that the solution of Eq. (1.1) blows up if the initial energy is nonpositive, i.e., $E[u_0] \le 0$, where the energy $E[u]$ is given by

$$E[u(t)] := \int |\nabla u(t)|^2 dx + \int |x|^2 |u(t)|^2 dx - \frac{1}{2} \int |u(t)|^4 dx. \tag{1.2}$$

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http://dx.doi.org/10.1016/j.aml.2016.06.007
0893-9659/

By using variational arguments, Zhang [19,20] investigated the sharp threshold for blow-up and global existence of the Eq. (1.1). In [21], the authors established a sharp threshold for blow-up solutions of (1.1) under the condition of finite initial energy, i.e., $0 < E[u_0] \leq \frac{3}{2^{\frac{5}{2}}}\|\varphi\|_2^2$, where the φ is the positive and spherically symmetric solution of the equation

$$-\Delta\psi + \psi - |\psi|^2\psi = 0, \quad \psi \in H^1(\mathbb{R}^3) \tag{1.3}$$

(see, e.g., [22,23]). Therefore, a natural but interesting question arises: Are there any blow-up solutions with arbitrary large initial energy? This will be the focus of the present paper.

In this paper, we are interested in studying the existence of blow-up solutions of the Cauchy problem (1.1). By establishing an invariant set and applying the Gagliardo–Nirenberg type inequality, we derive a new theorem on the existence of blow-up solutions with arbitrary large initial energy.

This paper is organized as follows. The forthcoming section is some preliminaries. In Section 3, we establish an invariant set. At the end, due to the invariant set and the Gagliardo–Nirenberg type inequality, we show the existence of blow-up solutions with arbitrary large initial energy in Section 4.

2. Preliminaries

Throughout this paper, we define the energy space in the course of nature as

$$H := \left\{ f \in H^1(\mathbb{R}^3) | \int |x|^2|f|^2dx < +\infty \right\}.$$

Here and hereafter, for simplicity, we use $\int \cdot dx$ denotes $\int_{\mathbb{R}^3} \cdot dx$. H is a Hilbert space, continuously embedded in $H^1(\mathbb{R}^3)$, when endowed with the inner product

$$\langle f, g\rangle_H := \int [\nabla f \nabla \bar{g} + f\bar{g} + |x|^2 f\bar{g}]dx,$$

whose associated norm we denote by $\|\cdot\|_{\mathrm{H}}$. Furthermore, we use $\|\cdot\|_p$ to denote the norm of $L^p(\mathbb{R}^3)$.

By Oh [14], we have the following local well-posedness for the Cauchy problem (1.1).

Lemma 2.1. *Let $u_0 \in H$. Then there exists a unique solution $u(x,t)$ of the Cauchy problem (1.1) in $C([0,T), H)$ for some $T \in (0, +\infty)$, either $T = +\infty$ or $T < +\infty$ and $\lim_{t\to T} \|u(t,\cdot)\|_H = \infty$. Moreover, we have conversation of mass*

$$M[u(t)] := \int |u(t)|^2dx = \int |u(0)|^2dx = M[u_0] \tag{2.1}$$

and conversation of energy

$$E[u(t)] = E[u_0]. \tag{2.2}$$

In addition, by a direct calculation (see T. Cazenave [15]; Glassy [9]) we have

Lemma 2.2. *Let $u_0 \in H$ and $u(x,t)$ be a solution of the Cauchy problem (1.1) on $[0,T)$. We put the variance $J(\cdot)$ by*

$$J(t) := \int |x|^2|u|^2dx. \tag{2.3}$$

Then the following identities hold:

$$J'(t) = 4Im \int x\bar{u}\nabla u dx, \tag{2.4}$$

$$J''(t) = 8\int |\nabla u|^2dx - 8\int |x|^2|u|^2dx - 6\int |u|^4dx. \tag{2.5}$$