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Revisit the over-relaxed proximal point algorithm[☆]

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ABSTRACT

The main purpose of this paper is to revisit the proximal point algorithms with over-relaxed A-maximal m-relaxed monotone mappings for solving variational inclusions in Hilbert spaces without Lipschitz continuity requirement to overcome the drawbacks of the paper (Verma, 2009) [5]. We affirmatively answer the open question mentioned in the paper (Huang and Noor, 2012) [6].

1. Introduction

Let X be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. We hope to solve the following general class of nonlinear variational inclusion problems: to find a solution to

$$0 \in M(x), \tag{1}$$

where $M : X \to 2^X$ is a set-valued mapping on X. Readers may refer to [1–17] for details.

The purpose of this paper is to revisit the proximal point algorithms for finding solution of (1) in Hilbert spaces. The proximal point algorithm could be traced back to [18], which was not capable of solving variational inclusions. In the light of Rockafellar's work [19,20], where its original purpose was to solve constrained nonlinear optimization problems, currently we are able to use the proximal point algorithm to study variational inclusion problems. However, ′the situation becomes considerably more complicated when M fails to monotone′. (See [3, Page 1081].) Some new approaches of the subject were taken in [3–6], dealing with a class of nonmonotone operators. Among these endeavors, Verma [5] provided a proximal

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point algorithm for over-relaxed A-maximal m-monotone inclusion problems, but unfortunately leading to incorrect results in [5]. We point out in [6] that the convergence rate of the proximal point algorithm [5] is larger than the real number one, implying that the strong convergence in [5] cannot be obtained accordingly. Up to date, the following question is still open: '*The question on whether the strong convergence holds or not for the over-relaxed proximal point algorithm is still open.*' (See [6, Page 1743].) To remedy the incorrectness of [5], in Section 3 we will suggest to use error analysis technique to analyze convergence. The strong convergence of this paper requires only the over-relaxed A-maximal m-monotonicity and continuity of the underlying operator. Therefore we affirmatively answer the above open question in [6].

2. Preliminaries

In this section we recall some important concepts and known results so as to prove the strong convergence of a new proximal point algorithm in Section 3.

Let $M : X \to 2^X$ be a multi-valued mapping on X. We denote both the mapping M and its graph by M, i.e., the set $\{(x, y) : y \in M(x)\}$. This is to state that a mapping is any subset M of $X \times X$, and $M(x) = \{y|(x, y) \in M\}$. If M is single valued, we will use $M(x)$ to represent the unique v such that $(x, v) \in M$. The domain of a map M is defined by

$$dom(M) := \{x \in X|\exists y \in X, \text{ such that } (x, y) \in M\} = \{x \in X|M(x) \neq \emptyset\}.$$

An inverse M^{-1} of M is defined by $\{(y, x)|(x, y) \in M\}$.

Definition 2.1 (*[3–5]*)**.** Let $M : X \to 2^X$ be a multi-valued mapping on X. The mapping M is said to be:

(i) Monotone if

$$\langle u^* - v^*, u - v\rangle \geq 0, \quad \forall (u, u^*), (v, v^*) \in graph(M).$$

(ii) r-strongly monotone if there exists a positive constant r such that

$$\langle u^* - v^*, u - v\rangle \geq r\|u - v\|^2, \quad \forall (u, u^*), (v, v^*) \in graph(M).$$

(iii) m-relaxed monotone if there exists a positive constant m such that

$$\langle u^* - v^*, u - v\rangle \geq (-m)\|u - v\|^2, \quad \forall (u, u^*), (v, v^*) \in graph(M). \tag{2}$$

Clearly a r-strongly monotone mapping must be a monotone mapping, and a monotone mapping must be a m-relaxed monotone mapping, but the converse is not true. Therefore the class of the m-relaxed monotone mappings defined in Definition 2.1(iii) is the most general class, and hence Definition 2.1(iii) includes all of Definition 2.1(i) and (ii) as special cases.

Definition 2.2 (*[5]*)**.** Let $A : X \to X$ be a single-valued mapping. The mapping $M : X \to 2^X$ is said to be *A-maximal m-relaxed monotone* if:

(i) M is m-relaxed monotone for $m > 0$,
(ii) $R(A + \rho M) = X$ for $\rho > 0$.