Applied Mathematics Letters 62 (2016) 118–123

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

On power-law fluids with the power-law index proportional to the pressure

J. Málek[a,*], K.R. Rajagopal[b], J. Žabenský[c]

[a] *Charles University in Prague, Faculty of Mathematics and Physics, Mathematical Institute, Sokolovská 83, 186 75 Prague 8, Czech Republic*

[b] *Department of Mechanical Engineering, Texas A&M University, College Station, TX 77843, USA*

[c] *University of Würzburg, Institute of Mathematics, Chair of Mathematics XI, Campus Hubland Nord, Emil-Fischer-Straße 40, 97074 Würzburg, Germany*

ARTICLE INFO

Article history:
Received 18 May 2016
Received in revised form 5 July 2016
Accepted 6 July 2016
Available online 14 July 2016

Keywords:
Power-law fluids
Pressure-dependent viscosity
Unsteady flows
Existence theory
Fluid dynamics

ABSTRACT

In this short note we study special unsteady flows of a fluid whose viscosity depends on both the pressure and the shear rate. Here we consider an interesting dependence of the viscosity on the pressure and the shear rate; a power-law of the shear rate wherein the exponent depends on the pressure. The problem is important from the perspective of fluid dynamics in that we obtain solutions to a technologically relevant problem, and also from the point of view of mathematics as the analysis of the problem rests on the theory of spaces with variable exponents. We use the theory to prove the existence of solutions to generalizations of Stokes' first and second problem.

1. Introduction

It is well established that in practically all fluids, the viscosity of the fluid can depend on the pressure (provided the pressure range is particularly large, see Bridgman [1], Szeri [2]) and in a wide class of fluids, viscosity can also depend on the shear rate. The viscosity of certain fluids can vary by as much as 10^8 due to variations in the pressure[1] (see Bair and Kottke [4] and the various references in a survey by Málek and Rajagopal [5] or in the later paper [6]) and it can change by orders of magnitude with respect to the shear rate. While there has been considerable work concerning the flows of fluids with pressure-dependent viscosity

* Corresponding author.
E-mail addresses: malek@karlin.mff.cuni.cz (J. Málek), krajagopal@tamu.edu (K.R. Rajagopal), josef.zabensky@mathematik.uni-wuerzburg.de (J. Žabenský).

[1] To be more precise, the mean value of the stress; see the recent paper by Rajagopal [3] for a detailed discussion of the notion of pressure.

http://dx.doi.org/10.1016/j.aml.2016.07.007

as well as those with a shear-rate-dependent viscosity, there has been very little work concerning the response of fluids whose viscosity depends on both the pressure and the shear rate simultaneously. Additionally, in these few studies that are devoted to the viscosity depending on the shear rate and the pressure, dependence of the viscosity on the shear rate is of the power-law type with the power-law exponent being a fixed number and they address primarily mathematical questions concerning the existence and uniqueness of solutions (see Franta et al. [7], Málek et al. [8], Bulíček et al. [9–11]) and are not concerned with the solution of specific initial–boundary value problems. Moreover, the dependence of the viscosity on both the pressure and the shear rate has to fulfill certain mathematical conditions which contradict the experimental observation that the viscosity tends to infinity with the pressure tending to infinity. There are a few numerical studies (see [12–15]) that consider the dependence of the viscosity on the pressure and the shear rate. However, the effect of the pressure and the shear rate on the viscosity are in the form of a product of a term representing the effect of the pressure and another term, representing the effect of the shear rate. It is, however, possible that the two influences on the viscosity cannot be so decomposed. This is indeed the situation in most instances, where more than one quantity can influence the properties of the material, for instance the temperature and the shear rate. Electrorheological fluids represent a very interesting situation wherein the viscosity of the fluid depends on both the shear rate and the electric field, with the dependence being expressed by the shear rate raised to the power of the electrical field (see Růžička [16]). Such a situation takes into consideration not only a physical possibility but it also opens up an interesting area in mathematical analysis (see Diening et al. [17]).

In this short note, we consider fluids whose viscosity depends on the pressure as well as the shear rate, with the variation that is similar to that discussed in the papers on electrorheology cited above in that the viscosity depends on the shear rate that is raised to a power that depends on the pressure (see Eq. (3)) while in electrorheological fluids the shear rate is raised to the intensity of the applied electric field. Our study extends the seminal studies of two unsteady problems considered by Stokes for a Newtonian fluid that are popularly referred to as Stokes' first problem (see Stokes [18]) and Stokes' second problem (see Stokes [18], Rayleigh [19]), in which Stokes considers a fluid above a plane, flowing due to the oscillation of the plane, and the problem of the flow of a Newtonian fluid lying above a plane, due to the plane being accelerated suddenly. Stokes did not take into account the effect of gravity. Srinivasan and Rajagopal [20] extended Stokes' study to take into account the effect of gravity as well as the pressure-dependence of the viscosity. As the pressure changes with depth, due to gravity, the viscosity of the fluid changes with depth and this gives rise to interesting physical consequences in that the vorticity and the shear stresses at the wall differ markedly from what one expects in a Newtonian fluid.

Recently, Rajagopal, Saccomandi and Vergori [21] studied unidirectional unsteady flows of fluids with pressure-dependent viscosity, where the effects of gravity are taken into account. After discussing the qualitative properties of the governing equations and establishing uniqueness for such unidirectional flows, they found explicit exact solutions for generalizations of Stokes' first and second problem for a special case of pressure-dependence of the viscosity, namely an exponential dependence of the viscosity on the pressure that obeys what is popularly referred to as the Barus formula (see Barus [22]). Rajagopal, Saccomandi and Vergori [21] do not consider the possibility of the viscosity depending simultaneously on both the pressure and the shear rate, which is the subject matter of this note.

The governing partial differential equations (6)–(8) and the initial–boundary conditions (5) pose quite a challenging problem that requires us to appeal to results in the theory of Lebesgue and Sobolev spaces with variables exponents (see Diening et al. [17]) in order to establish the existence of a weak solution to the governing equations.

Let $\Omega \subset \mathbb{R}^3$ be a three-dimensional domain (i.e. an open, connected set). In Ω, we consider unsteady flows of an incompressible, homogeneous fluid with a constant (strictly positive) density ϱ. The velocity $\boldsymbol{v} = (u, v, w)$ and the mean normal stress (pressure) p satisfy the equations representing the balance of linear