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Diffusive and inviscid traveling waves of the Fisher equation and nonuniqueness of wave speed

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ABSTRACT

In this paper we present an intuitive explanation for the non-uniqueness of the traveling wave speed in the Fisher equation, showing a similar non-uniqueness property in the case of inviscid traveling waves. More precisely, we prove that traveling waves of the Fisher equation with wave speed $c > 0$ converge to the inviscid traveling wave with speed $c > 0$ as the diffusion vanishes. A complete diagram that shows the relation between the diffusive and inviscid traveling waves is given in this paper.

1. Introduction

It is believed that traveling wave phenomena in a reaction diffusion equation,

$$u_t = du_{xx} + \psi(u), \quad u \geq 0,\ x \in \mathbf{R},$$

are obtained by an interplay between the diffusion and the reaction. For example, there exists a unique traveling wave solution for a bistable nonlinearity case, say $\psi(u) = u(1-u)(u-a)$, $0 < a < 1$, that connects the two stable steady states, $u = 0$ and 1. However, such a traveling wave solution does not exist if $d = 0$ or $\psi = 0$. In other words, the unique traveling wave solution has been *produced* by an interplay between the two different mechanisms. However, such a belief fails when the traveling wave connects a stable steady state to an unstable one. First of all, there exist inviscid ($d = 0$) traveling waves for any wave speed which stand without any help of diffusion. On the other hand, diffusive (or viscous) traveling waves exist only when the wave speed is greater than or equal to a minimum speed. In other words, the diffusion does not produce

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traveling waves, but a gap in wave speeds. The purpose of this note is to clarify the role of each component involved in the traveling wave phenomenon that connects a stable steady state to an unstable one.

To be more specific we consider the Fisher equation case in this note, i.e., $\psi(u) = u(1-u)$, where the stable steady state is $u = 1$ and the unstable one is $u = 0$. This Fisher equation provides the phenomenon in a simplest form. If $d = 0$, there is a traveling wave for any given wave speed $c > 0$ (see Section 2). We denote it by v_c and call it an *inviscid traveling wave*. However, if $d \neq 0$, there exists a traveling wave of a speed $c > 0$ if and only if $c \geq c^* := 2\sqrt{d}$. We denote it by $u_{c,d}$ and call it a *diffusive traveling wave*. The theory of this paper is for the relation between the inviscid and diffusive traveling waves. First we show that $u_{c,d} \to v_c$ uniformly as $d \to 0$ in Theorem 1. This convergence gives an insight for the existence of a continuum of traveling wave speeds in the Fisher equation, where each fixed speed corresponds to an inviscid traveling wave as $d \to 0$. The convergence $u_{c,d} \to u_{c^*,d}$ as $c \to c*$ with a fixed d is given in Theorem 2 and the convergence of $u_{c^*(d),d}$ to a step function as $d \to 0$ is given in Theorem 3. The convergence of v_c to the same step function as $c \to 0$ directly comes from the explicit formula of v_c in (2.5). Finally, these relations of convergence among the step function, diffusive and inviscid traveling waves complete a diagram of convergence given in Fig. 1, which is discussed in Section 4.

Studies of the vanishing viscosity limit are classical in hyperbolic problems of conservation laws. Such studies include the Fisher equation as a special case of zero convection when a monostable reaction term is included (see [1–5]). In particular, it was shown in [6–8] that the vanishing viscosity limit of minimum wave speeds is the minimum wave speed of the inviscid traveling wave, which is a related result to Theorem 3. The convergence relations given in this paper, Fig. 1, may provide a succinct insight for the full dynamics in a simplest form without convection.

Studies of the traveling wave phenomenon of reaction diffusion equations have a long history. The Fisher equation has been introduced by Fisher [9] and by Kolmogorov, Petrovskii and Piscounoff [10]. The purpose of Fisher was to perform modeling in population genetics, where the traveling wave solutions represented the spread of the advantageous gene through space. Later on, the Fisher–KPP equation was also used in ecology to model waves of an invading population (cf. Holmes et al. [11]) and in wound healing, where the solutions represent healing waves of cells in the skin (cf. Sherratt and Murray [12]).

2. Inviscid traveling waves

Consider the Fisher equation,

$$u_t = du_{xx} + u(1-u), \qquad u(x,0) = u^0(x), \quad t > 0,\ x \in \mathbf{R},$$

where $u(x,t)$ is a population density and $d > 0$ is a constant diffusivity. Let $z = x - ct$ be the variable for the traveling wave solution with a speed $c > 0$. Due to the symmetric structure of the equation we may consider positive wave speed $c > 0$ and the negative one can be treated symmetrically. It is well known that, for any $c \geq c^*$, there exists a traveling wave solution of wave speed c, where the minimum wave speed is $c^* = 2\sqrt{d}$ (see [13,14]). Let u be the traveling wave solution for a wave speed $c \geq c^*$, i.e., $u(x,t) = u(x-ct) = u(z)$. Since $u_t = -cu'$ and $\frac{\partial^2}{\partial x^2} u = u''$, the traveling wave solution satisfies

$$du'' + cu' + u(1-u) = 0, \quad z \in \mathbf{R}. \tag{2.1}$$

We are looking for a monotone traveling wave that connects the stable steady state $u = 1$ and the unstable one $u = 0$:

$$\lim_{z\to-\infty} u(z) = 1, \quad u(0) = 0.5, \quad \text{and} \quad \lim_{z\to\infty} u(z) = 0. \tag{2.2}$$

The conditions at infinity allow positive wave speeds only. Remember that the traveling wave phenomenon of the Fisher equation has translation invariance and the condition $u(0) = 0.5$ picks the symmetric one with