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General decay estimates for the wave equation with acoustic boundary conditions in domains with nonlocally reacting boundary

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ABSTRACT

In this paper, we deal with the general decay estimates for the wave equation with acoustic boundary conditions in domains with nonlocally reacting boundary.

1. Introduction

In this paper, we are concerned with the uniform decay rates of solutions for the wave equation:

$$\begin{cases} u'' - \Delta u + \rho(u') = 0 & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0, +\infty), \\ \dfrac{\partial u}{\partial \nu} = z' & \text{on } \Gamma_1 \times (0, +\infty), \\ u' + f(x)z'' - c^2 \Delta_\Gamma z + g(x)z' + h(x)z = 0 & \text{on } \Gamma_1 \times (0, +\infty), \\ u(x,0) = u_0(x), \qquad u'(x,0) = u_1(x) & \text{in } \Omega, \\ z(x,0) = z_0(x) & \text{on } \Gamma_1, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geq 2)$ with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint with $meas(\Gamma_0) > 0$, and ν represents the outward normal to Γ. $'$ denotes the derivative with respect to time t, Δ and Δ_Γ are the spatial Laplace and Laplace–Beltrami operators, respectively.

The models with acoustic boundary conditions have been widely investigated (see [1–5] and a list of references therein). Recently, Frota et al. [6] studied the uniform stability of wave equation with the boundary

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$(1.1)_3$–$(1.1)_4$, which is called the acoustic boundary conditions to nonlocally reacting boundary. This is a new physical formulation to the acoustic conditions. Since then, the problem with nonlocally reacting boundary has been studied by some authors (cf. [7–9]). However, all of the mentioned references showed the exponential or polynomial decay rates.

Motivated by previous works, the goal of this paper is to prove the general decay estimates for the wave equation with acoustic boundary conditions in domains with nonlocally reacting boundary. This paper is organized as follows: In Section 2, we give the hypotheses to prove our main result and introduce the energy decay theorem. In Section 3, we prove the general decay rates of (1.1).

2. Preliminaries

We begin this section introducing some notations and our main result. Throughout this paper we define the Hilbert space $\mathcal{H} = \{u \in H^1(\Omega); \Delta u \in L^2(\Omega)\}$ with the norm $\|u\|_{\mathcal{H}} = \left(\|u\|^2_{H^1(\Omega)} + \|\Delta u\|_2^2\right)^{\frac{1}{2}}$. Moreover, $L^p(\Omega)$-norm and $L^p(\Gamma)$-norm are denoted by $\|\cdot\|_p$ and $\|\cdot\|_{p,\Gamma}$, respectively, and $(u,v) = \int_\Omega u(x)v(x)dx$, $(u,v)_\Gamma = \int_\Gamma u(x)v(x)d\Gamma$. Denoting $\gamma_0 : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ and $\gamma_1 : \mathcal{H} \to H^{-\frac{1}{2}}(\Gamma)$ the trace map of order zero and the Neumann trace map on $\mathcal{H}$, respectively, we have $\gamma_0(u) = u_{|\Gamma}$ and $\gamma_1(u) = (\frac{\partial u}{\partial \nu})_{|\Gamma}$ for all $u \in D(\bar{\Omega})$. We denote $W = V \cap H^3(\Omega)$, where $V = \{u \in H^1(\Omega); \gamma_0(u) = 0 \text{ on } \Gamma_0\}$. By Poincaré's inequality, the norm $\|u\|_V = \left(\sum_{i=1}^n \int_\Omega (\frac{\partial u}{\partial x_i})^2 dx\right)^{\frac{1}{2}}$ is equivalent to the usual norm from $H^1(\Omega)$. Since Γ_1 is a compact manifold without boundary, it is possible to use norms in the spaces $H^1(\Gamma_1)$ and $H^2(\Gamma_1)$ by using the tangential gradient and the Laplace–Beltrami operator, respectively. Indeed, we consider the space $H^1(\Gamma_1)$ endowed with the norm $\|z\|^2_{H^1(\Gamma_1)} = \|z\|^2_{2,\Gamma_1} + \|\nabla_T z\|^2_{2,\Gamma_1}$, where ∇_T is the tangential gradient, and $H^2(\Gamma_1)$ equipped with the norm $\|z\|^2_{H^2(\Gamma_1)} = \|z\|^2_{2,\Gamma_1} + \|\Delta_\Gamma z\|^2_{2,\Gamma_1}$.

Now we give the hypotheses for the main result.

($\mathbf{H_1}$) Hypotheses on Ω.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here Γ_0 and Γ_1 are closed and disjoint with $meas(\Gamma_0) > 0$, satisfying the following condition:

$$m \cdot \nu \geq \sigma > 0 \quad \text{on } \Gamma_1, \;\; m \cdot \nu \leq 0 \text{ on } \Gamma_0, \quad m(x) = x - x^0 (x^0 \in \mathbb{R}^n), \tag{2.1}$$

where ν represents the unit outward normal vector to Γ.

($\mathbf{H_2}$) Hypotheses on ρ.

Let $\rho : \mathbb{R} \to \mathbb{R}$ be a nondecreasing C^1 function such that $\rho(0) = 0$ and suppose that there exists a strictly increasing and odd function β of C^1 class on $[-1, 1]$ such that

$$|\beta(s)| \leq |\rho(s)| \leq |\beta^{-1}(s)| \quad \text{if } |s| \leq 1, \tag{2.2}$$

$$C_1|s| \leq |\rho(s)| \leq C_2|s| \quad \text{if } |s| > 1, \tag{2.3}$$

where β^{-1} denotes the inverse function of β and C_1, C_2 are positive constants.

($\mathbf{H_3}$) Hypotheses on f, g, h.

Assume that f, g, h are essentially bounded satisfying $f(x), g(x), h(x) > 0$ for $x \in \Gamma_1$, that is, there exist positive constants f_i, g_i, h_i $(i = 0, 1)$ such that

$$f_0 \leq f(x) \leq f_1, \qquad g_0 \leq g(x) \leq g_1, \qquad h_0 \leq h(x) \leq h_1 \quad \text{for all } x \in \Gamma_1. \tag{2.4}$$

Considering above hypotheses, we have the following result: Assume that (H_1)–(H_3) hold and let $(u_0, u_1, z_0) \in W \times V \times H^2(\Gamma_1)$. Then there exists a unique pair of function (u, z) in the class

$$u \in L^\infty_{loc}(0, \infty; V \cap H^2(\Omega)), \qquad u' \in L^\infty_{loc}(0, \infty; V), \qquad u'' \in L^\infty_{loc}(0, \infty; L^2(\Omega)),$$
$$z \in L^\infty_{loc}(0, \infty; H^2(\Gamma_1)), \qquad z' \in L^\infty_{loc}(0, \infty; H^1(\Gamma_1)), \qquad z'' \in L^\infty_{loc}(0, \infty; L^2(\Gamma_1)),$$

that is a solution to the problem (1.1). The proof of the above existence result can be obtained by Faedo–Galerkin method (see [7,6]) and we will omit it.