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Stabilization via homogenization

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ABSTRACT

In this short note we treat a 1 + 1-dimensional system of changing type. On different spatial domains the system is of hyperbolic and elliptic type, that is, formally, $\partial_t^2 u_n - \partial_x^2 u_n = \partial_t f$ and $u_n - \partial_x^2 u_n = f$ on the respective spatial domains $\bigcup_{j\in\{1,\ldots,n\}} \left(\frac{j-1}{n}, \frac{2j-1}{2n}\right)$ and $\bigcup_{j\in\{1,\ldots,n\}} \left(\frac{2j-1}{2n}, \frac{j}{n}\right)$. We show that $(u_n)_n$ converges weakly to u, which solves the exponentially stable limit equation $\partial_t^2 u + 2\partial_t u + u - 4\partial_x^2 u = 2(f + \partial_t f)$ on $[0, 1]$. If the elliptic equation is replaced by a parabolic one, the limit equation is *not* exponentially stable.

1. Introduction

For $n \in \mathbb{N}$ and a given smooth f, we consider the following equation of mixed type:

$$\begin{cases} \partial_t^2 u_n(t,x) - \partial_x^2 u_n(t,x) = \partial_t f(t,x), & x \in \displaystyle\bigcup_{j\in\{1,\ldots,n\}} \left(\frac{j-1}{n}, \frac{2j-1}{2n}\right), \\ u_n(t,x) - \partial_x^2 u_n(t,x) = f(t,x), & x \in \displaystyle\bigcup_{j\in\{1,\ldots,n\}} \left(\frac{2j-1}{2n}, \frac{j}{n}\right), \qquad (t \in \mathbb{R}), \\ (\partial_x u_n)(t,0) = (\partial_x u_n)(t,1) = 0, \end{cases}$$

subject to zero initial conditions and conditions of continuity at the junction points $\{(2j-1)/2n; j \in \{1, \ldots, n-1\}\}$ for u_n. We show that for $n \to \infty$ the sequence of solutions $(u_n)_{n\in\mathbb{N}}$ converges weakly in $L^2_{\mathrm{loc}}(\mathbb{R} \times [0, 1])$ to u, which solves

$$\frac{1}{2}\partial_t^2 u(t,x) + \partial_t u(t,x) + \frac{1}{2}u(t,x) - 2\partial_x^2 u(t,x) = f(t,x) + \partial_t f(t,x), \quad ((t,x) \in \mathbb{R} \times (0,1)) \tag{1}$$

subject to $\partial_x u(t,0) = \partial_x u(t,1) = 0$ for $t \in \mathbb{R}$ and zero initial conditions. Moreover, we show that the asymptotic limit admits exponentially stable solutions. Note that the stability result for the limit equation

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is due to the superposed effect of the hyperbolic type and the elliptic type equation: Indeed, it is remarkable that $(\partial_t^2 - \partial_x^2)u = \partial_t f$ is *not* exponentially stable, if considered on the whole of $[0, 1]$ as underlying spatial domain. Moreover, we will show that if we replace the elliptic part, $u_n(t, x) - \partial_x^2 u_n(t, x) = f(t, x)$, by a corresponding parabolic one, that is, $\partial_t u_n(t, x) - \partial_x^2 u_n(t, x) = f(t, x)$ the limit equation reads

$$\partial_t^2 u(t, x) + \partial_t u - 2\partial_x^2 u(t, x) = f(t, x) + \partial_t f(t, x), \quad ((t, x) \in \mathbb{R} \times (0, 1)) \tag{2}$$

subject to homogeneous Neumann boundary conditions. Moreover, we find that the limit equation is *not* exponentially stable (in the sense of [1, Definition 3.1], see also [2, Section 3.1]).

For the proof of the homogenization (i.e. the computation of the limit equation) and stability results, we will employ the notion of evolutionary equations developed in [3,4]. We will use results on exponential stability of [1] (with an improvement in [5]) developed in this line of reasoning. The computation of the limit equation is based on [6,7]. In the next section, we will recall the notion of evolutionary equations and the results mentioned. The third section establishes the functional analytic framework for the equations to study. Moreover, we provide the proof of the result mentioned concerning the hyperbolic–elliptic system. We address the case where the parabolic equation replaces the elliptic one in the last section.

2. Evolutionary equations

In the whole section, let $\mathcal{H}$ be a Hilbert space. For $\nu \in \mathbb{R}$ we define

$$L_\nu^2(\mathbb{R}; \mathcal{H}) := \left\{ f\colon \mathbb{R} \to \mathcal{H}; f \text{ measurable}, \int_{\mathbb{R}} \|f(t)\|_{\mathcal{H}}^2 e^{-2\nu t} dt < \infty \right\}$$

endowed with the obvious norm (and scalar product). We set

$$\partial_{t,\nu}\colon D(\partial_{t,\nu}) \subseteq L_\nu^2(\mathbb{R}; \mathcal{H}) \to L_\nu^2(\mathbb{R}; \mathcal{H}), f \mapsto f',$$

where f' denotes the distributional derivative and $D(\partial_{t,\nu})$ is the maximal domain in $L_\nu^2(\mathbb{R}; \mathcal{H})$. Note that for all $\nu \neq 0$, we have $\partial_{t,\nu}^{-1}$ is a bounded linear operator in $L_\nu^2(\mathbb{R}; \mathcal{H})$, see [8, Corollary 2.5]. Note that also $\partial_{t,\nu}^{-1} f = \int_{-\infty}^{(\cdot)} f(\tau) d\tau$ for $f \in L_\nu^2(\mathbb{R}; \mathcal{H})$ and $\nu > 0$.

For a closed, densely defined linear operator B in $\mathcal{H}$, we shall denote the corresponding lifted operator to $L_\nu^2(\mathbb{R}; \mathcal{H})$ by the corresponding calligraphic letter, that is,

$$\mathcal{B}\colon L_\nu^2(\mathbb{R}; D(B)) \subseteq L_\nu^2(\mathbb{R}; \mathcal{H}) \to L_\nu^2(\mathbb{R}; \mathcal{H}), f \mapsto (t \mapsto Bf(t)).$$

The exponentially weighted L^2-type spaces have been used to obtain a solution theory for abstract operator equations in space time. $L(\mathcal{H})$ denotes the space of bounded linear operators in $\mathcal{H}$.

Theorem 2.1 *([3, Solution Theory], [4, Theorem 6.2.5]). Let A be a skew-self-adjoint operator in $\mathcal{H}$, $0 \leqslant M = M^*, N \in L(\mathcal{H})$. Assume there exist $c, \nu > 0$ such that for all $\mu \geqslant \nu$, we have*

$$\mu\langle M\varphi, \varphi\rangle + \mathrm{Re}\langle N\varphi, \varphi\rangle \geqslant c\langle \varphi, \varphi\rangle \quad (\varphi \in \mathcal{H}). \tag{3}$$

Then the operator $\mathcal{B}_\mu := \partial_{t,\mu}\mathcal{M} + \mathcal{N} + \mathcal{A}$ with $D(\mathcal{B}_\mu) = D(\partial_{t,\mu}) \cap D(\mathcal{A})$ is closable in $L_\mu^2(\mathbb{R}; \mathcal{H})$. Moreover, $\mathcal{S}_\mu := \overline{\mathcal{B}}_\mu^{-1}$ is well-defined, continuous and bounded with $\|\mathcal{S}_\mu\|_{L(L_\mu^2)} \leqslant 1/c$.

Remark 2.2. In the situation of Theorem 2.1, assume there is $\eta \in \mathbb{R}$ with the property that $\overline{\mathcal{B}}_\mu^{-1}|_{C_c^\infty(\mathbb{R};\mathcal{H})}$ extends to a bounded linear operator $\mathcal{S}_\zeta \in L(L_\zeta^2(\mathbb{R}; \mathcal{H}))$ for all $\zeta \geqslant \eta$. Then, by [4, Theorem 6.1.4] or [1, Lemma 3.6], for all $\zeta, \xi \geqslant \eta$ we have that $\mathcal{S}_\xi = \mathcal{S}_\zeta$ on $L_\zeta^2(\mathbb{R}; \mathcal{H}) \cap L_\xi^2(\mathbb{R}; \mathcal{H})$.