Applied Mathematics Letters 60 (2016) 115–119

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

A lower bound for the blow-up time to a viscoelastic hyperbolic equation with nonlinear sources[☆]

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ARTICLE INFO

Article history:
Received 24 January 2016
Accepted 24 March 2016
Available online 6 April 2016

Keywords:
Viscoelastic hyperbolic equation
Energy estimate method
Lower bound estimate

ABSTRACT

This paper deals with the lower bound for blow-up solutions to a nonlinear viscoelastic hyperbolic equation. An inverse Hölder inequality with the correction constant is employed to overcome the difficulty caused by the failure of the embedding inequality. Moreover, a lower bound for blow-up time is obtained by establishing first-order differential inequality. This result gives an answer to the problem unsolved in our earlier work Sun et al. (2014).

1. Introduction

In this paper, we study the following hyperbolic equation with linear damping

$$\begin{cases} u_{tt} - \triangle u - \omega\triangle u_t + \mu u_t = |u|^{p-2}u & \text{in } [0,T]\times\Omega, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } [0,T]\times\partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in R^N. $\mu > 0, p > 2$.

We refer to [1] for the motivation and references concerning the study of Problem (1.1). It is well known that the source term causes finite-time blow-up of solutions [2–6]. However, it is natural to ask that, if the solution blows up in finite time, can we give an estimate of a lower bound to the blow-up time? In fact, we all know that the upper bound ensures blowing-up of the solution and the importance of the lower bound is that it may provide us a safe time interval for operation if we use Problem (1.1) to model a physical process such as viscoelastic fluids, processes of filtration through a porous media, fluids with temperature-dependent viscosity etc. But, in general, it is very hard to obtain a lower bound estimate for hyperbolic problems

☆ The first author's research was supported by NSFC (11301211) and NSF of Jilin Prov.(201500520056JH).

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http://dx.doi.org/10.1016/j.aml.2016.03.017
0893-9659/

because the method that one estimates the derivative of the control functional to establish a first-order differential inequality in parabolic cases fails. In particular, the first author, Sun and Gao in [7] applied an energy estimate method and the Sobolev inequalities to give an estimate of the lower bound for the blow-up time when $2 < p \leqslant \frac{2(N-1)}{N-2}$. Therefore, we find when the exponent p lies in the interval $(\frac{2(N-1)}{N-2}, \frac{2N}{N-2}]$, the embedding relationship $H_0^1(\Omega) \hookrightarrow L^{2p-2}(\Omega)$ does not hold, which leads to that our method used in [7] is no longer effective. In order to overcome this difficulty, we have to develop some new ideas or techniques. In this paper, we apply the interpolation inequality and an energy estimate method to prove the inverse Hölder inequality with correction constant (Lemma 1.2) and then construct the suitable control functional to establish a differential inequality. Before stating our main result, let us recall some results on the existence and blow-up of the solution of Problem (1.1).

Theorem 1.1 ([5]). *Let u be the unique local solution to (1.1). If the following conditions are satisfied*

$$
\begin{aligned}
&(H_1)\ u_0 \in H_0^1(\Omega), \quad u_1 \in L^2(\Omega),\\
&(H_2)\ \omega > 0, \quad \mu > -\omega\lambda_1,\\
&(H_3)\ 2 < p \leq \frac{2N}{N-2},\ N \geqslant 3; \quad \text{or } 2 < p < \infty, \quad N = 1,\ \text{or } 2,
\end{aligned}
$$

λ_1 is the first eigenvalue of the operator $-\triangle$ under homogeneous Dirichlet boundary condition. Then $T^* < \infty$ if and only if there exists a $\bar{t} \in [0, T^*)$ such that

$$u(\bar{t}) \in U \quad \text{and} \quad E(u(\bar{t}), u_t(\bar{t})) \leq d,$$

where

$$
\begin{aligned}
&T^* = \sup\{T > 0; u = u(t) \text{ exists on } [0, T]\},\\
&U = \{u \in H_0^1(\Omega); J(u) \leq d, I(u) < 0\} \quad \text{and} \quad E(u, v) = J(u) + \frac{1}{2}\|v\|_2^2,\\
&J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p, \quad I(u) = \|\nabla u\|_2^2 - \|u\|_p^p,\\
&d = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \max_{\lambda \geq 0} J(\lambda u).
\end{aligned}
$$

By Theorems 1.3 and 2.1 in [7], we have the following results about an estimate to the upper and lower bounds.

Theorem 1.2 ([7]). *Assume that all the conditions of Theorem 1.1 hold and $\int_\Omega u_0 u_1 dx > 0$, then there exists a positive number $T_1 = T_1(\|u_0\|_2, \omega, \mu, \|u_1 u_0\|_1) < \infty$ such that $T^* \leqslant T_1$. That is*

$$\lim_{t \to T^{*-}} \|u\|_{L^p(\Omega)} = +\infty.$$

Theorem 1.3 ([7]). *Assume that all the conditions of Theorem 1.1 hold, and*

$$u_0 \in U, \quad \int_\Omega u_0 u_1 dx > 0, \quad E(u_0, u_1) \leq d, \quad 2 < p \leq \frac{2(N-1)}{N-2} \quad \text{for } N \geq 3.$$

Then the blow-up time T^* satisfies the following estimate

$$T^* \geq \int_{H(0)}^{\infty} \frac{1}{C_3 y^q + 2y + C_4} dy,$$

where the constants C_3, C_4 depend on $p, N, |\Omega|$ and $E(0)$. $q = \begin{cases} 3 - \frac{4}{p}, & 2 < p \leqslant \frac{2N}{N-1}; \\ p - 1, & \frac{2N}{N-1} < p \leqslant \frac{2(N-1)}{N-2}. \end{cases}$ and $H(0) = \int_\Omega |u_0|^p dx$.