Applied Mathematics Letters 61 (2016) 1–7

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Decay rate of solutions to 3D Navier–Stokes–Voigt equations in H^m spaces

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ARTICLE INFO

Article history:
Received 23 February 2016
Received in revised form 27 April 2016
Accepted 27 April 2016
Available online 6 May 2016

Keywords:
Navier–Stokes–Voigt equations
Decay rate
Fourier Splitting Method
Inductive argument

ABSTRACT

In this paper, we first prove the regularity in $H^m(\mathbb{R}^3)$ of weak solutions to the Navier–Stokes–Voigt equations with initial data in $H^K(\mathbb{R}^3)$ for all $m \leq K$. Then we compute the upper bound of decay rate for these solutions, specifically, we prove that

$$\|\nabla^m u(t)\|^2 + \|\nabla^{m+1} u(t)\|^2 \leq c(1+t)^{-3/2-m}, \quad \text{for large } t,$$

when $u_0 \in H_\sigma^{m+1}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, $m \in \mathbb{N}$.

1. Introduction

In this paper we consider the following 3D Navier–Stokes–Voigt equations

$$\begin{cases} u_t - \alpha^2 \Delta u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p = 0, & x \in \mathbb{R}^3, t > 0, \\ \nabla \cdot u = 0, & x \in \mathbb{R}^3, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, α is a length scale parameter characterizing the elasticity of the fluid and u_0 is the initial velocity. The system (1.1) was first used by Oskolkov [1] to study the motion of certain viscoelastic incompressible fluids and has been proposed by Cao, Lunasin and Titi [2] as a regularization, for small value of α, of the 3D Navier–Stokes equations for the sake of direct numerical simulations.

In the last few years, the existence and long-time behavior of solutions to the Navier–Stokes–Voigt equations has attracted the attention of many mathematicians. In bounded domains or unbounded domains

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http://dx.doi.org/10.1016/j.aml.2016.04.015

satisfying the Poincaré inequality, there are many results on the existence and long-time behavior of solutions in terms of existence of attractors for Navier–Stokes–Voigt equations, see e.g. [3–6] and references therein. In the whole space, Zhao and Zhu [7] have recently proved the existence of weak solutions to (1.1) and more importantly, computed the decay rate for these solutions, that is,

$$\|\nabla^m u(t)\|^2 + \|\nabla^{m+1} u(t)\|^2 \le c(1+t)^{-3/2-m}, \quad \text{for large } t, \tag{1.2}$$

when $u_0 \in H^{m+1}_\sigma(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, but only for $m = 0$, 1. See also a recent paper [8] for decay characterization of solutions in terms of the initial datum. As mentioned in [7, Remark 3.1], extension of the above result on decay rate of solutions to the case of general nonnegative integer m is an interesting open problem.

The aim of this paper is to give the affirmative answer for this question. More precisely, we extend the result of Zhao and Zhu in [7] to the $H^m(\mathbb{R}^3)$-norms for all $m \in \mathbb{N}$. To do this, we follow the general lines of the approach used by Bjorland and Schonbek for the viscous Camassa–Holm equations in [9], by combining the Fourier Splitting Method with an inductive argument. The Fourier Splitting Method was built up by Schonbek in [10–12] for the decay rate of solutions to Navier–Stokes equations and then developed for other equations.

Denote $L^p(\mathbb{R}^3) = (L^p(\mathbb{R}^3))^3, 1 \le p \le \infty$, with the norm

$$\|u\|_{L^p} = \left(\int_{\mathbb{R}^3} |u(x)|^p dx\right)^{1/p}; \qquad \|u\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}^3} |u(x)|,$$

particularly $\|\cdot\|_{L^2} := \|\cdot\|$, and

$$W^{m,p}(\mathbb{R}^3) = \left\{u = (u_1, u_2, u_3) \in L^p(\mathbb{R}^3) \mid D^\beta u \in L^p(\mathbb{R}^3),\ |\beta| \le m\right\}$$

with the norm

$$\|u\|_{m,p} := \left(\sum_{k=1}^{3} |u_k|^p_{m,p}\right)^{1/p}, \quad \text{where } |u_k|_{m,p} := \left(\int_{\mathbb{R}^3}\left(\sum_{|\beta|\le m} |D^\beta u_k|^p\right)\right)^{1/p}.$$

Especially, we denote $H^m(\mathbb{R}^3) = W^{m,2}(\mathbb{R}^3), \|\cdot\|_{m,2} := \|\cdot\|_{H^m}$ and by $H^m_\sigma(\mathbb{R}^3)$ the closure of $\{u \in (C^\infty_0(\mathbb{R}^3))^3 \mid \nabla \cdot u = 0\}$ with respect to the $H^m(\mathbb{R}^3)$-norm.

The rest of the paper is organized as follows. In Section 2, we prove the regularity of weak solutions for Eqs. (1.1). The decay rate for these weak solutions is computed in the last section. As is explained in [7, Remark 3.1], it suffices to give formal calculations in the proofs below.

2. The regularity of weak solutions

First, we recall the definition and result on the existence and uniqueness of weak solutions to the Navier–Stokes–Voigt equations (1.1).

Definition 2.1 *([7])***.** A function $u \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; H^1_\sigma(\mathbb{R}^3))$ is called a weak solution of Eqs. (1.1) if it satisfies

$$\int_0^\infty \int_{\mathbb{R}^3} \left(-u\cdot\partial_t\psi - \alpha^2 u\cdot\partial_t\Delta\psi - \nu u\cdot\Delta\psi + (u\cdot\nabla)u\cdot\psi\right) dx\, dt = \int_{\mathbb{R}^3} u_0\cdot\psi(x,0)dx,$$

for all test functions $\psi \in C^\infty_0(\mathbb{R}^3\times\mathbb{R}_+, \mathbb{R}^3)$ with $\nabla\cdot\psi = 0$.

Theorem 2.1 *([7]).* *For $u_0 \in H^1_\sigma(\mathbb{R}^3)$ given, Eqs. (1.1) have a unique weak solution. If furthermore $u_0 \in H^2_\sigma(\mathbb{R}^3)$, then the weak solution u has the following regularity*

$$u \in L^\infty(\mathbb{R}_+; H^1_\sigma(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; H^2_\sigma(\mathbb{R}^3)),\ \partial_t u \in L^2(\mathbb{R}_+; L^2(\mathbb{R}^3)).$$