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Shape-dependent natural boundary condition of Lagrangian field

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ABSTRACT

In the framework of the Lagrangian field theory, we derive the equations characterizing shape-dependent natural boundary conditions from the Hamilton's principle. Of these equations, one exhibits mathematical pattern similar to general relativity. In this equation, one side of the sign of equality is the energy–momentum tensor of field and another side is the combination of mean curvature and Gaussian curvature of boundary surface. Meanwhile, we verify that the shape-dependent natural boundary condition can be simplified into the shape equation of lipid vesicle or the generalized Young–Laplace's equation under different condition.

1. Introduction

Conventionally, the boundary value problem of a field $\varphi(t, \mathbf{x})$ consists of partial differential equations satisfied by $\varphi(t, \mathbf{x})$ as well as boundary conditions that are imposed *ad hoc* based on physical considerations at the boundary points. The boundary conditions are characterized by $\varphi(t, \mathbf{x})$, the derivative of $\varphi(t, \mathbf{x})$ or combination of $\varphi(t, \mathbf{x})$ and the derivative of $\varphi(t, \mathbf{x})$. Generally, they are supposed to be independent of curvature of boundary surface. However, if surface energy is concerned, the mean curvature of boundary surface has been taken into account in the boundary conditions. Such boundary conditions are referred to as the surface-energy-dependent boundary conditions. Some typical examples can be found in capillary wave, phase transition and bio-membrane.

The surface-energy-dependent boundary conditions can be traced back to Young and Laplace's works on capillary surface [1], in which they proposed the so-called Young–Laplace's equation. Taking this equation as the traction boundary condition, the Lord Rayleigh gave a solution to the oscillation of spherical droplet [2]. This solution was later used to model the oscillation and fission of nucleus [3]. Gurtin and Murdoch extended the Young–Laplace equation into the generalized Young–Laplace equation so as to characterize the surface

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of elastic solid [4]. Further, Steigmann and Ogden proposed reinforced boundary condition by taking into account the bending stiffness of the surface film [5]. Zhu, Ru and Chen discussed non-uniqueness of boundary value problems based on the generalized Young–Laplace equation [6].

Recently, Javili, dell'Isola and Steigmann formulated a geometrically nonlinear theory of higher-gradient elasticity accounting for boundary energies [7]. Based on analytical continuum mechanics, Auffray, dell'Isola and Eremeyev et al. derived the Euler–Lagrange equations and boundary conditions for second gradient continua in terms of an objective deformation energy volume density [8]. Eremeyev and Altenbach advanced the kinematical compatibility conditions characterizing interaction of a second-gradient fluid with an elastic solid with the surface elasticity [9]. In these works, a common feature is that the surface energy density depends only on first- and second-gradient of deformation, not explicitly containing the mean and Gaussian curvature. This means that no coupling exists between deformation and the shape of boundary surface.

As a traction boundary condition, the generalized Young–Laplace equation has been applied to investigate physical behaviors of nano-structured materials. On the relevant literature, the reader can refer to the reviews by Wang et al. [10] and Sun [11]. When the Young–Laplace's equation is used as the traction boundary condition, the mean curvature in the equation is considered to be known in general. This means that the shape of boundary surface is given in advance. However, some recent works have shown that nonlinear coupling exists between the field (e.g., electrostatic field, chemical concentration field and so on) in soft matter and the shape of its boundary surface [12]. Under this case, the shape of boundary surface is unknown. Therefore, simultaneously with solving the boundary value problem of field, the shape of boundary surface also needs to be determined. Since the shape of boundary surface is influenced by the field, a key problem is how to characterize the coupling between the field and the shape of its boundary surface. To the best of my knowledge, this is a problem awaiting to be explored.

The aim of this paper is to characterize the coupling between the field and the shape of its boundary surface in the framework of the Lagrangian field theory. As will be seen later, this coupling represents a type of new natural boundary condition. We call it the shape-dependent natural boundary condition.

The paper is outlined as follows. In Section 2, we introduce a curvature-dependent surface energy density in the action functional of field. From this action functional, the Lagrangian equation and shape-dependent natural boundary condition are derived. In Section 3, we verify that the shape-dependent natural boundary condition can be simplified into the shape equation of lipid vesicle or the generalized Young–Laplace's equation under different condition. Finally, we summarize and comment on the results in this paper.

Notation: The index rules and summation convention are adopted. Latin indices run from 1 to 3. The Greece letter Ω stands for a bounded domain of R^3, and $\partial\Omega$ is the boundary surface of Ω. The derivatives with respect to coordinates are represented as $\partial_k = \partial/\partial x_k$ or $(\cdot)_{,k} = \partial(\cdot)/\partial x_k$. $\partial^k = g^{kj}\partial_j$, where g^{kj} is the metric tensor. The derivative with respect to time is denoted by an upper dot, e.g., $\dot{a} = \mathrm{d}a/\mathrm{d}t$. Other symbols will be introduced in the text where they appear for the first time.

2. Lagrangian field with flexible boundary surface

Let $\mathbf{x} = \{x^j\}$ $(j = 1, 2, 3)$ be a 3-dimensional position vector in Ω and $t \in [t_0, t_1]$ be time. A physical field defined on $[t_0, t_1] \cup \Omega$ is denoted by $\varphi = \varphi(t, \mathbf{x})$. Depending on circumstances, φ can be a scalar, vector or tensor field. The Lagrangian of the field φ is written as $L = L(\varphi, \dot{\varphi}, \partial_j\varphi)$.

Let spatial domain Ω occupied by φ be bounded and the surface $\partial\Omega$ of Ω be a smooth surface. We believe that physical behaviors of φ in the interior of Ω is different from that on the boundary of Ω. Therefore, the action functional of φ is supposed to have the form below

$$A = \int_{t_0}^{t_1}\int_{\Omega} L(\varphi, \dot{\varphi}, \partial_j\varphi)\mathrm{d}v(\mathbf{x})\mathrm{d}t - \int_{t_0}^{t_1}\int_{\partial\Omega} \gamma(\varphi, \partial_j\varphi, H, K)\mathrm{d}a(\mathbf{x})\mathrm{d}t, \tag{1}$$