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A high order numerical scheme for variable order fractional ordinary differential equation☆

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ABSTRACT

In this paper, we derive a high order numerical scheme for variable order fractional ordinary differential equation by establishing a second order numerical approximation to variable order Riemann–Liouville fractional derivative. The scheme is strictly proved to be stable and convergent with second order accuracy, which is higher than some recently derived schemes. Finally, some numerical examples are presented to demonstrate the theoretical analysis and verify the efficiency of the proposed method.

1. Introduction

In the past few decades, fractional calculus has been widely used to describe the process of memory and hereditary properties in various fields of science and engineering [1,2]. As a main mathematical tool, fractional differential equations have received tremendous success both in theoretical and numerical aspects [3–6]. However, when people try to use fractional diffusion equations to model some complex systems, they find that many physical processes appear to exhibit anomalous diffusion behavior may vary with time, or/and space, those processes can't be well characterized by constant order fractional diffusion equations. Then the concept of variable order operator has been introduced [7–12]. But the analytical solutions of variable order fractional differential equations are still difficult to obtain, so developing numerical methods with high order accuracy is urgent and important. Up to now, limited work has been done in this field [13–18]. However, the

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numerical accuracy was just first order. This motivates us to further study high order numerical methods for variable order fractional operators. The main contribution of this paper is that we establish a high order numerical scheme for variable order fractional ordinary differential equation by constructing a second order numerical approximation to variable order Riemann–Liouville fractional derivative.

Consider the following variable order fractional differential equation

$$\begin{cases} {}_C D_{0,t}^{\alpha(t)} y(t) = f(t), \quad 0 \le t \le T, \\ y(0) = 0, \end{cases} \tag{1}$$

where ${}_C D_{0,t}^{\alpha(t)}$ is the variable order Caputo fractional derivative, $0 < \alpha_{\min} \le \alpha(t) \le \alpha_{\max} < 1$. The fractional derivative ${}_C D_{0,t}^{\alpha(t)} y(t)$ is defined by [9]

$${}_C D_{0,t}^{\alpha(t)} y(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_0^t (t-s)^{-\alpha(t)} y'(s) ds, \tag{2}$$

where $\Gamma(\cdot)$ denotes Gamma function.

Remark 1. Without loss of generality, we assume that the initial condition $y(0) = 0$. If $y(0) = \mu$, one can consider the equation for $v(t) = y(t) - \mu$ instead.

The remainder of this paper is organized as follows. In Section 2, the derivation of a second order numerical method for solving Eq. (1) is proposed. Stability and convergence analysis of the proposed scheme are established in Section 3. Numerical examples are provided to demonstrate the theoretical analysis in Section 4. Finally, we summarize this paper in Section 5.

2. The derivation of the scheme

We first recall that the variable order Riemann–Liouville fractional derivative of order $\alpha(t) \in (0,1)$ is defined as [7,9]

$${}_{RL} D_{0,t}^{\alpha(t)} y(t) = \frac{1}{\Gamma(1-\alpha(t))} \frac{d}{dt} \int_0^t (t-s)^{-\alpha(t)} y(s) ds. \tag{3}$$

Similar to constant order fractional operators, if $y(t) \in \mathcal{C}[0,\infty)$, then we have the following relationship.

Lemma 1.

$${}_C D_{0,t}^{\alpha(t)} y(t) = {}_{RL} D_{0,t}^{\alpha(t)} \left[y(t) - y(0) \right]. \tag{4}$$

Proof. Integration by parts and differentiation, we complete the proof. □

Since the initial condition $y(0) = 0$ in (1), so ${}_C D_{0,t}^{\alpha(t)} y(t) = {}_{RL} D_{0,t}^{\alpha(t)} y(t)$. In order to construct a second order approximation to variable order Riemann–Liouville derivative, we introduce the following shifted Grünwald approximation [15] to variable order Riemann–Liouville derivative

$$\mathcal{A}_{\tau,p}^{\alpha(t)} y(t) = \frac{1}{\tau^{\alpha(t)}} \sum_{k=0}^{\infty} g_k^{\alpha(t)} y(t-(k-p)\tau), \tag{5}$$

where $g_k^{\alpha(t)} = (-1)^k \binom{\alpha(t)}{k}$ for $k \ge 0$. Motivated by [19], we develop a second order approximation to variable order Riemann–Liouville fractional derivative in the following way.