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Existence of weak solutions for fractional porous medium equations with nonlinear term

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ABSTRACT

We study the following fractional porous medium equations with nonlinear term

$$\begin{cases} u_t + (-\Delta)^{\sigma/2}(|u|^{m-1}u) + g(u) = h, & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & \text{in } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0, & \text{in } \Omega. \end{cases}$$

The authors in de Pablo et al. (2011) and de Pablo et al. (2012) established the existence of weak solutions for the case $g(u) \equiv 0$. Here, we consider the nonlinear term g is without an upper growth restriction. The nonlinearity of g leads to the invalidity of the Crandall–Liggett theorem, which is the critical method to establish the weak solutions in de Pablo et al. (2011) and de Pablo et al. (2012). In addition, because of g does not have an upper growth restriction, we have to apply the weak compactness theorem in an Orlicz space to prove the existence of weak solutions by using the Implicit Time Discretization method.

1. Introduction

In this paper, we consider the existence of weak solutions for the following fractional porous medium equations:

$$\begin{cases} u_t + (-\Delta)^{\sigma/2}(|u|^{m-1}u) + g(u) = h(x), & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subseteq \mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$, $u_0 \in L^{m+1}(\Omega)$, $h(x) \in L^\infty(\Omega)$, $0 < \sigma < 2$ and $m > 1$. We assume that g satisfies the natural dissipative condition

$$g'(u) \geq -C + k|u|^{q-1}, \tag{1.2}$$

for some $q > 1$, and $k > 0$.

There is wide interest in porous medium equations, which replace the nonlocal diffusion operator $(-\Delta)^{\sigma/2}$ by the classical Laplacian $-\Delta$, see for instance [1,2] and references therein. Recently, the authors of the pioneering papers [3,4] investigate the following equation without nonlinear term:

$$\begin{cases} u_t + (-\Delta)^{\sigma/2}(|u|^{m-1}u) = 0, & \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ u(x,0) = f(x), & \text{in } \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $f(x) \in L^1(\mathbb{R}^N)$. Using the Crandall–Liggett theorem, they got the existence, uniqueness and regularity of the weak solution.

Because of g is nonlinear, the Crandall–Liggett theorem is invalid. We use a compactness method to get a weak convergence sequence u_ε, whose weak limit will give us the solution. However, since the absence of an upper growth restriction for g, it is impossible to estimate $g(u_\varepsilon)$ to determine the weak limit. In order to overcome the difficulty, we apply the weak compactness theorem in an Orlicz space [5], as it is used for instance in [6].

2. Some preliminaries

The fractional Laplacian operator $(-\Delta)^{\sigma/2}$ in a bounded domain is defined by a spectral decomposition [7–9,3,4]. Let $\{\varphi_k, \lambda_k\}_{k=1}^\infty$ be the eigenfunctions and the corresponding eigenvector of $-\Delta$ in Ω with Dirichlet boundary condition. The operator $(-\Delta)^{\sigma/2}$ is defined by any $u \in C_0^\infty(\Omega)$, $u = \Sigma_{k=1}^\infty u_k\varphi_k$,

$$(-\Delta)^{\sigma/2}u = \Sigma_{k=1}^\infty \lambda_k^{\sigma/2} u_k \varphi_k.$$

This operator can be extended by density for u in the Hilbert space

$$H_0^{\sigma/2}(\Omega) = \{u \in L^2(\Omega) : \|u\|^2_{H_0^{\sigma/2}} = \Sigma_{k=1}^\infty \lambda_k^{\sigma/2} u_k^2 < \infty\}.$$

Multiplying both sides of the equation in (1.1) by test function $\varphi \in C_0^1(\Omega \times (0,T))$ and integrating by parts, we obtain

$$\begin{aligned} &-\int_0^T \int_\Omega u\varphi_t dxds + \int_0^T \int_\Omega (-\Delta)^{\sigma/4}|u|^{m-1}u(-\Delta)^{\sigma/4}\varphi dxds \\ &+ \int_0^T \int_\Omega g(u)\varphi dxds = \int_0^T \int_\Omega h(x)\varphi dxds. \end{aligned} \tag{2.1}$$

Definition 2.1. A function is a weak solution of Problem (1.1) if $u \in L^\infty([0,\infty); L^{m+1}(\Omega))$, $|u|^{m-1}u \in L^2_{loc}((0,\infty); H_0^{\sigma/2}(\Omega))$; the identity (2.1) holds for every $\varphi \in C_0^1(\Omega \times (0,T))$ and $u(0,\cdot) = u_0$ holds almost everywhere in Ω.

The fractional Laplacian can be also defined by σ-harmonic extension which was introduced by Caffarelli and Silvestre for the case of the whole space in [10], and extended to bounded domains in [7,11], see also [12,9].

If $u(x)$ is a smooth bounded function defined on Ω, its extension to the upper half-cylinder $C_\Omega = \Omega \times (0,\infty)$, $U = E(u)$, is unique smooth bounded solution of the equation

$$\begin{cases} \nabla \cdot (y^{1-\sigma}\nabla U) = 0, & \text{in } C_\Omega, \\ U = 0, & \text{on } \partial\Omega \times [0,\infty), \\ U(x,0) = u(x), & \text{on } \Omega. \end{cases} \tag{2.2}$$