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Uniqueness for an inverse space-dependent source term in a multi-dimensional time-fractional diffusion equation

Ting Wei*, Liangliang Sun, Yushan Li

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730030, PR China

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ABSTRACT

This paper is devoted to identify a space-dependent source term in a multi-dimensional time-fractional diffusion equation from boundary measured data. The uniqueness for the inverse source problem is proved by the Laplace transformation method.

1. Introduction

Time-fractional diffusion equations are deduced by replacing the standard time derivative with a time fractional derivative and can be used to describe superdiffusion and subdiffusion phenomena [1–3]. The direct problems for time-fractional diffusion equations have been studied extensively in recent years [4,5]. However, the inverse problems for fractional diffusion equations have not been investigated widely. For uniqueness results, Cheng et al. [6] gave a uniqueness result for determining the order of fractional derivative and the diffusion coefficient in a time-fractional diffusion equation. Sakamoto and Yamamoto in [5] established some uniqueness results for several inverse problems. Yamamoto et al. in [7] provided a conditional stability in determining a zeroth-order coefficient in a half-order fractional diffusion equation.

In this paper we consider the uniqueness of an inverse space-dependent source problem for a multi-dimensional time fractional diffusion equation in a general bounded domain.

Let Ω be a bounded domain in $\mathbb{R}^d$ with sufficient smooth boundary $\partial\Omega$. Consider the following time-fractional diffusion equation

$$\partial_{0+}^{\alpha} u(x,t) = \Delta u(x,t) + f(x)p(t), \quad x \in \Omega,\ t \in (0,T), \tag{1.1}$$

* Corresponding author.
E-mail address: tingwei@lzu.edu.cn (T. Wei).

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where ∂_{0+}^{α} for $0 < \alpha < 1$ denotes the Caputo fractional left derivative of order α with respect to t

$$\partial_{0+}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha}}, \quad 0 < \alpha < 1,\ t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function and $T > 0$ is a fixed final time or $+\infty$.

Suppose the unknown function u satisfies the following initial and boundary value conditions

$$u(x,0) = \phi(x), \quad x \in \overline{\Omega}, \tag{1.2}$$

$$\frac{\partial u}{\partial n}(x,t) = 0, \quad x \in \partial\Omega,\ t \in (0,T), \tag{1.3}$$

where n is the unitary normal vector, exterior to the domain Ω.

If all functions $f(x)$, $p(t)$, $\phi(x)$ are given appropriately, the problem (1.1)–(1.3) is a direct problem. The inverse problem here is to determine the source term $f(x)$ based on problem (1.1)–(1.3) and additional data

$$u(x,t)\,|_{\Gamma} = g(x,t), \quad 0 < t < T, \tag{1.4}$$

where Γ is a nonempty open part of $\partial\Omega$.

For one-dimensional case with $p(t) \equiv 1$, Zhang and Xu in [8] have proved a uniqueness result for reconstructing the space-dependent source function $f(x)$ by using the Cauchy data at one end $x = 0$ and provided a numerical method to obtain the stable approximation. For two-dimensional case, Li and Guo in [9] gave a uniqueness result for $p(t) \equiv 1$ with the additional data at $x = (0,0)$ located on a corner of a square domain. However the proofs in the papers mentioned above cannot be used directly for the problem with $p(t) \not\equiv 0$ and the uniqueness does not hold for an irregular solution domain just using the measured data at a point since the eigenvalues of the operator $-\Delta$ with homogeneous Neumann boundary condition may not be simple.

In this paper, we focus on a multi-dimensional problem in a general domain. The uniqueness for determining the space-dependent source term $f(x)$ by measurement data on a part of boundary is given.

2. Uniqueness for the inverse source problem

Denote the eigenvalues of $-\Delta$ with homogeneous Neumann boundary condition as λ_n and the corresponding eigenfunctions as $\varphi_n \in \{\psi \in H^2(\Omega); \frac{\partial\psi}{\partial n}|_{\partial\Omega} = 0\}$, that means we have $\Delta\varphi_n = -\lambda_n\varphi_n$. Since $-\Delta$ is a symmetric uniformly elliptic operator, counting according to the multiplicities, we can set: $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$ and $\{\varphi_n\}$ is an orthonormal basis in $L^2(\Omega)$. Define the Hilbert scale space

$$\mathcal{D}\left((-\Delta+1)^{\gamma}\right) = \left\{\psi \in L^2(\Omega); \sum_{n=0}^{\infty} (\lambda_n+1)^{2\gamma} |(\psi,\varphi_n)|^2 < \infty\right\}, \tag{2.1}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, and define its norm

$$\|\psi\|_{\mathcal{D}((-\Delta+1)^{\gamma})} = \left(\sum_{n=0}^{\infty} (\lambda_n+1)^{2\gamma} |(\psi,\varphi_n)|^2\right)^{\frac{1}{2}}. \tag{2.2}$$

If $T > 0$ is finite, let $AC[0,T]$ be the space of absolutely continuous functions on $[0,T]$. Based on the method used in [5], we can obtain a strong solution $u \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega))$ such that $\partial_{0+}^{\alpha} u \in C((0,T]; L^2(\Omega)) \cap L^2((0,T)\times\Omega)$ under the conditions $\phi \in \mathcal{D}\left((-\Delta+1)^{\frac{1}{2}}\right)$, $f \in L^2(\Omega)$, $p \in AC[0,T]$ given by

$$u(x,t) = \sum_{n=0}^{\infty} (\phi,\varphi_n) E_{\alpha,1}(-\lambda_n t^{\alpha}) \varphi_n(x) + \sum_{n=0}^{\infty} (f,\varphi_n) p_n(t) \varphi_n(x), \tag{2.3}$$