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Recovering the initial distribution for space-fractional diffusion equation by a logarithmic regularization method

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ABSTRACT

In this paper, the backward problem for space-fractional diffusion equation is investigated. We proposed a so-called logarithmic regularization method to solve it. Based on the conditional stability and an *a posteriori* regularization parameter choice rule, the convergence rate estimates are given under *a-priori* bound assumption for the exact solution.

1. Introduction

In this article, we consider the following backward problem for the diffusion equation with space fractional Laplacian

$$u_t(x,t) + (-\Delta)^{\alpha} u(x,t) = 0, \quad x \in \mathbb{R},\ t \in (0,+\infty), \tag{1.1}$$

$$u(x,T) = f^{\delta}(x), \quad x \in \mathbb{R},\ u(x,t)|_{x\to\pm\infty} = 0, \tag{1.2}$$

where $\frac{1}{2} < \alpha \leq 1$, and the one-dimensional fractional Laplacian is defined pointwise by the principal value integral [1]

$$(-\Delta)^{\alpha}\phi(x) = c_{\alpha} P.V. \int_{\mathbb{R}} \frac{\phi(x) - \phi(y)}{|x-y|^{1+2\alpha}} dy, \tag{1.3}$$

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here $c_\alpha = \frac{2^{2\alpha}\alpha\Gamma(\frac{1}{2}+\alpha)}{\sqrt{\pi}\Gamma(1-\alpha)}$. Moreover, the Fourier transform of fractional Laplacian is given by [1]

$$\widehat{(-\Delta)^\alpha\phi}(\omega) = |\omega|^{2\alpha}\widehat{\phi}(\omega). \tag{1.4}$$

The backward problem is determining the initial distribution $u(x,0) = \varphi(x)$ from the final value measurement data $f^\delta(x)$. In fact, according to the Fourier transform, it is easy to see that the fractional Laplacian is the symmetric case (skewness $\theta = 0$) of the Riesz–Feller fractional derivative ${}_xD^\alpha_\theta$, which is defined by Mainardi, Luchko and Pagnini in [2]. Such fractional derivative has a wide range of applications in thermoelasticity [3], ecology [4], and finance, especially modeling for the high-frequency price dynamics in financial markets [5]. It is worth noting that the non-locality of fractional Laplacian and the ill-posedness lead to the main difficulties for solving the backward problem (see [6]). Therefore, in order to overcome the difficulties, the common way is to introduce a appropriate regularization method. For example, in [6], Zheng and Wei proposed a convolution regularization method and spectral regularization method to solve the backward problem. Furthermore, Shi et al. presented an *a posteriori* parameter choice rule for the convolution regularization method and deduced a log-type error estimate in [7]. In [8], Cheng et al. proposed an iteration regularization method to deal with the inverse problem. A simplified Tikhonov regularization method is applied by Zhao et al. to solve it in [9].

In this paper, we present a logarithmic regularization method to solve the backward diffusion problem. That is, we introduce the following variational functional with a logarithmic type penalty term

$$J(\varphi) = \frac{1}{2}\|u(\varphi(x);x,T) - f^\delta(x)\|^2 + \frac{\beta}{2}\left\|\left[\mathcal{F}^{-1}\left(\ln(1+(1+|\omega|^2)^{\alpha_1})\right) * \varphi\right](x)\right\|^2, \tag{1.5}$$

where "$*$" denotes the convolution operation, $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, $\alpha_1 \in \mathbb{R}$ is a arbitrary constant, and $\beta \in (0,1)$ is a regularization parameter. The minimizer of (1.5) is defined as the regularization solution.

2. Convergence rate estimate of logarithmic regularization method

The Fourier transform and inverse Fourier transform of function $f(x)$ are respectively written as

$$\hat{f}(\omega) = \mathcal{F}\{f(x);\omega\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx,$$
$$f(x) = \mathcal{F}^{-1}\{\hat{f}(\omega);x\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i\omega x}d\omega,$$

$\|\cdot\|$ denotes the L^2 norm in $\mathbb{R}$, i.e.

$$\|f\| = \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^{\frac{1}{2}},$$

and $\|\cdot\|_{H^p(\mathbb{R})}$ denotes the H^p norm:

$$\|f\|_{H^p(\mathbb{R})} = \left(\int_{\mathbb{R}} (1+\omega^2)^p|\widehat{f}(\omega)|^2 d\omega\right)^{\frac{1}{2}}.$$

Through Fourier transform, the solution of backward problem (1.1)–(1.2) can be written in the form

$$\hat{u}(\omega,0) = e^{T|\omega|^{2\alpha}}\hat{f}^\delta(\omega). \tag{2.1}$$

Lemma 1 (*Conditional Stability [10]*)**.** *If* $\|u(\cdot,0)\|_{H^p(\mathbb{R})} \le E$ $(p>0)$, *then we have*

$$\|u(\cdot,0)\| \le \sqrt{2}ET^{\frac{p}{2\alpha}}\frac{1}{\left(\ln\frac{1}{\|f(\cdot)\|}\right)^{\frac{p}{2\alpha}}}. \tag{2.2}$$