Applied Mathematics Letters 61 (2016) 167–172

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Oscillation criteria for second order sublinear dynamic equations with oscillating coefficients[☆]

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ARTICLE INFO

Article history:
Received 9 May 2016
Received in revised form 2 June 2016
Accepted 2 June 2016
Available online 11 June 2016

Keywords:
Second order
Sublinear
Oscillation
Time scales

ABSTRACT

This paper is concerned with oscillation of second order sublinear dynamic equations with oscillating coefficients. By using generalized Riccati transformations, oscillation theorems are obtained on an **arbitrary** time scale. Our results provide new oscillation criteria for arbitrary time scales.

1. Introduction

Let $\mathbb{T}$ be a time scale (i.e., a nonempty closed set of the reals) and assume $\sup \mathbb{T} = \infty$. In this paper, we investigate oscillation of second order sublinear dynamic equations of the form

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta} + p(t)f(x^{\sigma}(t)) = 0, \quad t \in \mathbb{T},\ t \geq t_0. \tag{1.1}$$

Throughout this paper, we shall assume the following conditions hold:

(H_1) $p \in C_{rd}(\mathbb{T}, \mathbb{R})$, $1/r \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ and $\int_{t_0}^{\infty} \frac{1}{r(s)} \Delta s < \infty$;
(H_2) $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies $f'(x) > 0$ for $x \neq 0$, and $f(0) = 0$;
(H_3) the sublinear conditions: $0 < \int_0^{\varepsilon} \frac{1}{f(u)} du, \int_0^{-\varepsilon} \frac{1}{f(u)} du < +\infty$ for all $\varepsilon > 0$.

☆ This project was supported by the National Natural Science Foundation of China (No. 11271139 and 11271380).

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http://dx.doi.org/10.1016/j.aml.2016.06.005
0893-9659/

For background information on time scales, we refer to the monograph by Bohner and Peterson [1].

A nontrivial solution $x(t)$ of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its nontrivial solutions are oscillatory. The oscillation theory of dynamic equations has been developed extensively during the past several years. We refer the reader to the monographs [2–11] and the references cited therein.

For completeness, we review some earlier results. In 1975, Kusano and Naito [9] considered the following second-order sublinear differential equation

$$(r(t)x'(t))' + p(t)x^{\alpha}(t) = 0, \quad t \geq t_0 \tag{1.2}$$

and established the following necessary and sufficient conditions when $r(t) > 0$ and $\int_{t_0}^{\infty} \frac{1}{r(t)} dt < \infty$.

Theorem 1.1. *Let $0 < \alpha < 1$ be a quotient of odd positive integers, and $p(t) \geq 0$. Then Eq. (1.2) is oscillatory if and only if $\int_{t_0}^{\infty} R(s)p(s)ds = \infty$, where $R(s) = \int_{s}^{\infty} \frac{1}{r(\xi)} d\xi$.*

In 1993, Zhang [11] considered the following second order nonlinear difference equation

$$\Delta(r(n)\Delta x(n)) + p(n)x^{\alpha}(n+1) = 0, \quad t \in \mathbb{N} \tag{1.3}$$

and established the following necessary and sufficient conditions when $r(n) > 0$ and $\sum_{n=1}^{\infty} \frac{1}{r(n)} < \infty$.

Theorem 1.2. *Let $0 < \alpha < 1$ be a quotient of odd positive integers, and $p(n) \geq 0$. Then Eq. (1.3) is oscillatory if and only if $\sum_{n=1}^{\infty} R(n+1)p(n) = \infty$, where $R(n) = \sum_{i=n}^{\infty} \frac{1}{r(i)}$.*

It is of particular interest to consider Eq. (1.1) when the coefficient function $p(t)$ is allowed to be negative for arbitrarily large values of t. Based on this consideration, Jia et al. [3,7] established some sufficient conditions for oscillation of all solutions of Eq. (1.1). However, these results of Jia et al. [3,7] were obtained for the case that $\mathbb{T}$ is a regular time scale. That is, $\mathbb{T}$ is a time scale with $\inf \mathbb{T} = t_0$ and $\sup \mathbb{T} = \infty$, and $\mathbb{T}$ is either an isolated time scale (all points in $\mathbb{T}$ are isolated) or $\mathbb{T}$ is the real interval $[t_0, \infty)$.

2. Some lemmas

We shall need the following second mean value theorem (see [3,7]) and the differential inequality (see [10]), which will be useful in the proofs of the main results. We formulate these as Lemmas 2.1 and 2.2.

Lemma 2.1. *Let h be a bounded function that is integrable on $[a, b]$. Let m_H and M_H be the infimum and supremum, respectively, of the function $H(t) := \int_a^t h(s)\Delta s$ on $[a, b]$. Suppose that g is nonincreasing with $g(t) \geq 0$ on $[a, b]$. Then there is some number Λ with $m_H \leq \Lambda \leq M_H$ such that $\int_a^b h(t)g(t)\Delta t = g(a)\Lambda$.*

Lemma 2.2. *Assume $G : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}$. Assume further that $F : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. If the function F satisfies $F'(u) \geq 0$ and $F''(u) \leq 0$, then*

$$F'(G^{\sigma}(t))G^{\Delta}(t) \leq (F \circ G)^{\Delta}(t) \leq F'(G(t))G^{\Delta}(t).$$

3. Main results

Theorem 3.1. *Assume that conditions* (H$_1$)–(H$_3$) *hold. If there exists a function $\Phi > 0$ satisfying $\Phi^{\Delta} \leq 0$ and $\left(\Phi^{\Delta} r\right)^{\Delta} \geq 0$ such that*

$$\int_{t_0}^{\infty} \frac{1}{\Phi^{\sigma}(s)r(s)} \Delta s = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \Phi^{\sigma}(s)p(s)\Delta s = \infty,$$

then Eq. (1.1) is oscillatory.