Applied Mathematics Letters 57 (2016) 1–6

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

On the stability of homogeneous steady states of a chemotaxis system with logistic growth term

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ARTICLE INFO

Article history:
Received 2 October 2015
Received in revised form 2 December 2015
Accepted 3 December 2015
Available online 11 January 2016

Keywords:
Chemotaxis
Stability
Steady state
Lower and upper solutions

ABSTRACT

We consider a nonlinear PDEs system of Parabolic–Elliptic type with chemotactic terms. The system models the movement of a population "n" towards a higher concentration of a chemical "c" in a bounded domain Ω.

We consider constant chemotactic sensitivity χ and an elliptic equation to describe the distribution of the chemical

$$n_t - d_n \Delta n = -\chi \mathrm{div}(n\nabla c) + \mu n(1-n),$$
$$-d_c \Delta c + c = h(n)$$

for a monotone increasing and Lipschitz function h.

We study the asymptotic behavior of solutions under the assumption of $2\chi|h'| < \mu$. As a result of the asymptotic stability we obtain the uniqueness of the strictly positive steady states.

1. Introduction

Mathematical models of chemotaxis were introduced by Keller and Segel [1] in order to model the movement and aggregation of amoebae responding to a chemical stimulus. In the last four decades, chemotactic terms have been used to model different types of biological phenomena, as angiogenesis, morphogenesis, immune system response, etc. The specific model we are studying features a coupled system of two PDEs: a parabolic equation with a logistic growth term modeling the density of a population n,

$$n_t - d_n \Delta n = -\chi \mathrm{div}(n\nabla c) + \mu n(1-n) \quad \text{in } x \in \Omega,\ t \in (0,T), \tag{1.1}$$

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http://dx.doi.org/10.1016/j.aml.2015.12.001

where d_n, χ and μ are positive constants and $\Omega \subset \mathbb{R}^N$ is a bounded domain with regular boundary $\partial\Omega$. An elliptic equation describes the concentration of a chemical substance c, which acts as the chemoattractant:

$$-d_c \Delta c = h(n) - c \quad \text{in } x \in \Omega, \tag{1.2}$$

with initial datum and boundary conditions

$$\frac{\partial n}{\partial \mathbf{n}} = 0 \quad \text{on } x \in \partial\Omega,\ t \in (0, T), \tag{1.3}$$

$$n(0) = n_0 \quad \text{on } x \in \Omega, \tag{1.4}$$

$$\frac{\partial c}{\partial \mathbf{n}} = 0 \quad \text{on } x \in \partial\Omega. \tag{1.5}$$

Solutions to (1.1)–(1.5) which are biologically meaningful must satisfy

$$n \geq 0, \qquad c \geq 0. \tag{1.6}$$

The function h represents the production of the chemical substance by the living organisms, which, depending on the process, can take different forms. In the literature the function h has different representations:

- $h(n) = n$, see Jäger and Luckhaus [2] and Tello and Winkler [3]
- $$h(n) = s\frac{n}{\beta + n} \tag{1.7}$$

  where c satisfies a parabolic equation (see Orme and Chaplain [4])
- A polynomial function $h(n) = n^p$.

Myerscough et al. [5] study numerically the steady states of (1.1)–(1.5) and (1.7) focusing on the role of boundary conditions. In [5] the authors found non-constant steady states for a range of boundary conditions including (1.3) and (1.5). The parameters studied in [5] are not considered in Theorem 1.2.

We will study the problem (1.1)–(1.5), for a general function h satisfying

$$h \text{ is locally Lipschitz function}, \tag{1.8}$$

there exists a positive constant $\alpha > 0$ such that

$$0 \leq h' \leq \frac{\mu d_c}{2\chi}(1 - \alpha). \tag{1.9}$$

We also assume that the initial data $u_0 \in W^{2,p}(\Omega)$ for some $p > N$,

$$\frac{\partial u_0}{\partial \vec{n}} = 0 \quad \text{in } \partial\Omega$$

and there exist positive constants $\underline{n}_0$ and $\overline{n}_0$ such that

$$0 < \underline{n}_0 \leq n_0 \leq \overline{n}_0 < \infty. \tag{1.10}$$

In Section 2 we prove the following theorem.

Theorem 1.1. *Under assumptions (1.8)–(1.10), there exists a unique solution (n, c) to (1.1)–(1.5) and it satisfies*

$$|n - 1|_{L^\infty(\Omega)} + |c - h(1)|_{L^\infty(\Omega)} \longrightarrow 0 \quad \textit{when } t \longrightarrow \infty. \tag{1.11}$$

In Section 3 we shall use Theorem 1.1 to study the steady states of (1.1)–(1.5). The result is enclosed in the following theorem: