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Homogeneous initial–boundary value problem of the Rosenau equation posed on a finite interval

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ABSTRACT

This paper considers the IBVP of the Rosenau equation

$$\begin{cases} \partial_t u + \partial_t \partial_x^4 u + \partial_x u + u\partial_x u = 0, & x \in (0,1),\ t > 0, \\ u(0,x) = u_0(x) \\ u(0,t) = \partial_x^2 u(0,t) = 0, & u(1,t) = \partial_x^2 u(1,t) = 0. \end{cases}$$

It is proved that this IBVP has a unique global distributional solution $u \in C([0,T]; H^s(0,1))$ as initial data $u_0 \in H^s(0,1)$ with $s \in [0,4]$. This is a new global well-posedness result on IBVP of the Rosenau equation with Dirichlet boundary conditions.

1. Introduction

In this paper, we consider the initial–boundary-value problem (IBVP) of the Rosenau equation

$$\begin{cases} \partial_t u + \partial_t \partial_x^4 u + \partial_x u + u\partial_x u = 0, & x \in (0,1),\ t > 0, \\ u(0,x) = u_0(x) \\ u(0,t) = \partial_x^2 u(0,t) = 0, & u(1,t) = \partial_x^2 u(1,t) = 0. \end{cases} \tag{1.1}$$

The Rosenau equation was proposed in [1,2] when analyzing the discrete RLC circuit in the late 1980s. It is a generalization of the BBM equation (see [3])

$$\partial_t u - \partial_t \partial_x^2 u + \partial_x u + u\partial_x u = 0. \tag{1.2}$$

Till now, the classical solution and the distributive solution to IBVP of (1.2) have been shown to uniquely exist in [4,5]. However, IBVP (1.1) has only been proved to admit a unique classical solution (see [6,7]),

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although many works have been done from a numerical point of view (see [8]-[9], for example). Yet, it is still an unsolved problem that whether IBVP (1.1) admits a distributional solution as $u_0 \in H^s(0,1)$ with $0 \le s < 4$.

Motivated by [5,7], we will solve this problem in this paper. Our result is as follows.

Theorem 1.1 (*Global Well-posedness*)**.** *Given $0 < T < \infty$ and $u_0 \in H^s(0,1)$ with $s \in [0,4]$. The initial–boundary value problem (1.1) has a unique distributional solution*

$$u \in C([0,T]; H^s(0,1)).$$

Moreover, the solution mapping is locally Lipschitz continuous.

In order to prove Theorem 1.1, we first prove that IBVP (1.1) has a unique local solution as $s \in [0,4]$ by fixed point argument. Then we prove Theorem 1.1 holds for $s = 0$ and $s = 4$ by a priori estimates method. Finally we apply Tartar's interpolation theorem to extend the index to $(0,4)$.

Remark 1.2. The distributional solution to IBVP of (1.2) established in [4] needs $u_0 \in C(0,1)$, i.e. $u_0 \in H^s(0,1)$, $s > \frac{1}{2}$. However, for IBVP (1.1), the result is better since the distributional solution as $u_0 \in H^s(0,1)$ with $0 \le s \le \frac{1}{2}$ can be established.

Remark 1.3. Our global well-posedness result depends heavily on the homogeneous boundary conditions, while this result may not be suitable for the Rosenau equation with mixed boundary conditions as that in [10].

2. Linear IBVP of the Rosenau equation

In this section, we aim to prove that the linear IBVP

$$\begin{cases} \partial_t u + \partial_t \partial_x^4 u = \partial_x f, & x \in (0,1),\ t > 0, \\ u(x,0) = u_0(x), \\ u(0,t) = \partial_x^2 u(0,t) = 0, & u(1,t) = \partial_x^2 u(1,t) = 0. \end{cases} \tag{2.1}$$

has a unique solution $u \in C([0,T]; H^s(0,1))$ for any given $u \in H^s(0,1)$ with $s \in [0,4]$.

Proposition 2.1. *Given $0 \le s \le 4$ and $u_0 \in H^s(0,1)$. The IBVP (2.1) has a unique distributional solution u with the explicit form*

$$u(x,t) = u(x,0) + \int_0^t \int_0^1 g(x,\xi) \partial_\xi f(\xi,t) d\xi, \tag{2.2}$$

where $g(x,\xi)$ is Green's function associated with the fourth order elliptic equation

$$\begin{cases} v + \partial_x^4 v = 0, & x \in (0,1), \\ v(0) = \partial_x^2 v(0) = 0, & v(1) = \partial_x^2 v(1) = 0. \end{cases} \tag{2.3}$$

Proof. From Theorem 1 in [11], (2.3) has a unique solution $v(x) = 0$. Then applying Green's function formula (see [12], page 203, (3.3.5)), $g(x,\xi)$ can be constructed as

$$g(x,\xi) = \begin{cases} 2a_1(\xi) sh\left(\frac{-1+i}{\sqrt{2}} x\right) + 2a_2(\xi) sh\left(\frac{1+i}{\sqrt{2}} x\right), & 0 \le x < \xi, \\ A_1(\xi)\left(e^{\frac{-1+i}{\sqrt{2}}x} - e^{\frac{2(i-1)}{\sqrt{2}}} e^{\frac{1-i}{\sqrt{2}}x}\right) + A_2(\xi)\left(e^{\frac{1+i}{\sqrt{2}}x} - e^{\frac{2(1+i)}{\sqrt{2}}} e^{\frac{-1-i}{\sqrt{2}}x}\right), & \xi < x \le 1, \end{cases} \tag{2.4}$$