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A note on mixed boundary value problems involving triple trigonometrical series

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ABSTRACT

This study was motivated by the two-dimensional hydrodynamic slamming problem of a steep wave hitting a vertical wall. The fundamental problem considers dual impact on the wall at the lower and upper regions resembling the impact of a wave at the time of its breaking. The solution method results into a mixed-boundary value problem that involves a triplet of trigonometrical series which, to the author's best knowledge, has not been investigated in the past. The formulation of the mixed-boundary value problem is generic and could be used in different fields as well.

1. Introduction

Trigonometrical series that hold in different regions are often encountered in mixed-boundary value problems in potential theory. The vast majority of relevant studies consider dual trigonometrical series. On the contrary, triple trigonometrical series have not been investigated to the extent that dual trigonometrical series have been studied.

Looking back in the literature one could find that the first solution to dual series of this kind was given by Shepherd [1]. Analytical solutions have been given by Tranter [2–4]. The solutions provided in [2,3] were admittedly complicated and accordingly were simplified by the same author [4]. Srivastav [5] showed that certain dual trigonometrical relations can be reduced to a Fredholm integral equation of the second kind and under specific conditions can admit closed forms.

Studies approaching the analytic solution of triple trigonometrical series are rarely found in the literature. For example the classical book of Sneddon [6] has no reference and surprisingly the only book written after Sneddon's book on mixed-boundary value problems, that of Duffy [7], has only one specific example on sine

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series. Examples on triple trigonometrical series are the studies of Tranter [8] and Kerr et al. [9]. In Ref. [8] Tranter showed that the solution of triple trigonometrical series can be sought by solving an equivalent system of three integral equations. Although Tranter [8] considered both sine and cosine series involving harmonics $\sin[(n-1/2)\theta]$ and $\cos[(n-1/2)\theta]$, he did not consider the case when the argument $(n-1/2)$ that multiplies the expansion coefficients is reversed.

The present study is specifically dedicated to the analytic solution of mixed-boundary value problems involving triple trigonometrical series. The task of the present study is twofold. Firstly, to complement Tranter's [8] work on the transformation of mixed-boundary value problems involving triple trigonometrical series to associated problems involving triple integral equations and secondly, to provide an analytical solution for the concerned problem.

2. Triple trigonometrical series — Transformation to triple integral equations

The mixed-boundary value problem under investigation is composed of the following triplet of trigonometrical series:

$$\sum_{n=1}^{\infty} A_n \cos\left[(n-1/2)x\right] = 0 \quad (0 < x < a) \tag{1}$$

$$\sum_{n=1}^{\infty} (n-1/2)^{-1} A_n \cos\left[(n-1/2)x\right] = G(x) \quad (a < x < b) \tag{2}$$

$$\sum_{n=1}^{\infty} A_n \cos\left[(n-1/2)x\right] = 0 \quad (b < x < \pi). \tag{3}$$

Eqs. (1)–(3) can be considered as a special (homogeneous) case of the more general (non-homogeneous) triple series given for example in [10,11]. The sums of the left hand sides of (1) and (3) are expected to be singular at the boundaries $x = a$ and $x = b$. Our first goal is to show that this system can be transformed into a group of three integral equations. The sought system is widely referred in the literature as *triple integral equations of Titchmarsh type* [12]. Tranter's [8] method cannot be employed to this particular system as a straightforward application of it would require the analytical form of the integral $\int \sin\left[t \sin(x/2)\right] \mathrm{d}x$ which apparently is not known.

For the solution of the system (1)–(3) we start by letting $u = \sin(x/2)$ and accordingly $x = 2\arcsin(u)$ and we rely on the validity of the following expressions (see [13], p. 717, equation 6.671.2; p. 728, equation 6.693.2):

$$\int_0^{\infty} J_s(t)\cos(ut)\mathrm{d}t = \begin{cases} \dfrac{\cos\left[s \arcsin(u)\right]}{\sqrt{1-u^2}} & u < 1 \\ \infty & u = 1 \end{cases} \tag{4}$$

for $\mathrm{Re}(s) > -1$ and

$$\int_0^{\infty} t^{-1} J_s(t)\cos(ut)\mathrm{d}t = \frac{1}{s}\cos\left[s \arcsin(u)\right] \quad u \le 1. \tag{5}$$

In (4)–(5) J_s denotes the Bessel function of the first kind with order s.

Letting $s = 2n-1$ and introducing (4) into (1) and (3) and (5) into (2) we arrive at

$$\sum_{n=1}^{\infty} A_n \int_0^{\infty} J_{2n-1}(t)\cos(ut)\mathrm{d}t = 0 \quad [0 < u < \sin(a/2)] \tag{6}$$

$$\sum_{n=1}^{\infty} A_n \int_0^{\infty} t^{-1} J_{2n-1}(t)\cos(ut)\mathrm{d}t = \frac{1}{2} G\left[2\arcsin(u)\right] \quad [\sin(a/2) < u < \sin(b/2)] \tag{7}$$