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Shadow waves for pressureless gas balance laws

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ABSTRACT

The pressureless gas dynamic model with body force as a source is considered. The problem is solved using the procedure used for finding delta shock type solutions to a special conservation laws known as Shadow Waves. If the body force is interpreted as the acceleration constant multiplied by the density, the solution obtained in this paper looks physically reasonable since the velocities of waves are changed accordingly with the acceleration.

1. Introduction

The aim of this paper is to solve the pressureless gas dynamics model (PGD for short) with the added body force. That model can be derived from the well known isentropic gas dynamics model adding a force term on the right-hand side of momentum conservation law, and taking the pressure to be constant:

$$\begin{aligned} \partial_t\rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) &= b\rho. \end{aligned} \tag{1}$$

The force considered here is assumed to be the gravity with b being the gravity constant. It is known that the Riemann problem for conservation laws system as PGD can be solved uniquely by the delta shock solutions. One can look in [1] for the definition of measure valued solutions, in [2] for the sticky particles method, in [3] for the variational method. We expect that similar type of waves will appear in the above balance law (1). Among a lot of different approaches in explaining such type of solutions, we will use the one from [4], the so called shadow waves (SDW). Shadow waves are represented by nets of piecewise constant function

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with respect to space variable x, parametrized by some small parameter $\varepsilon > 0$ and bounded in $L^1_{loc}(\mathbb{R})$. The parameter ε enables us to include the Dirac delta function as a part of solution. Roughly speaking, we perturb a speed c of a wave from both sides by some small parameter ε so that left- and right-hand states are connected by a state that can be of order $1/\varepsilon$ in some components. The main advantage of the above defined states is that one uses only Rankine–Hugoniot conditions for each ε. So we obtain a net of classical weak solutions that satisfy the system in a distributional limit as $\varepsilon \to 0$. Also, the usual entropy inequality can be easily checked regardless of the form of entropy and entropy-flux functions. We shall use here a simpler condition—the so called overcompressibility which requires that characteristics should run into the shock curve. The further advantage of this approach is a simple treatment of the interaction problem involving a shadow wave. This will be commented at the end of this paper.

System (1) with $b = -1$ is given as a model of violent discontinuities in shallow water models with large Froude number in [5]. A solution containing the Dirac delta function appears there, too. In the paper [6], the author solves (1) using a change of coordinates that transfers it into a conservative one with flux depending on the time variable.

2. Elementary waves

Let us first state some known fact about elementary waves of the given system. One can look in [7] or [8] for more details. The system is weakly hyperbolic with the double eigenvalue $\lambda_{1,2} = u$. Let us first look for a solution to (1) when initial data are constants, $(\rho(x,0), u(x,0)) = (\rho_0, u_0)$. For smooth solutions, one can substitute ρ_t from the first equation of (1) into the second one and eliminate ρ from it by division (provided that we are away from a vacuum state). So, we have now the equation $\partial_t u + u\partial_x u = b$ that can be solved by the method of characteristics: $u = bt + u_0$, $x = x_0 + \frac{1}{2}bt^2 + u_0 t$. The first equation then becomes $\partial_t\rho + (bt + u_0)\partial_x\rho = 0$ with a solution $\rho = \rho_0$ on each curve $x = x_0 + \frac{1}{2}bt^2 + u_0 t$. So, the solution for constant initial data is $(\rho, u) = (\rho_0, bt + u_0)$, and it will be used in the rest of the paper.

Let us now look at the Riemann problem $(\rho,u)(x,0) = \begin{cases} (\rho_0, u_0), & x < 0 \\ (\rho_1, u_1), & x > 0. \end{cases}$ If $u_0 < u_1$, we have a solution of the form CD+Vacuum+CD—two contact discontinuities connected by the vacuum:

$$(\rho,u)(x,t) = \begin{cases} (\rho_0, u_0 + bt), & x < \dfrac{1}{2}bt^2 + u_0 t \\ (0, u), & \frac{1}{2}bt^2 + u_0 t < x < \frac{1}{2}bt^2 + u_1 t \\ (\rho_1, u_1 + bt), & x > \dfrac{1}{2}bt^2 + u_1 t \end{cases} \tag{2}$$

where u is an arbitrary function satisfying $u(bt^2/2 + u_0 t, t) = bt + u_0$ and $u(bt^2/2 + u_1 t, t) = bt + u_1$.

3. Shadow waves

In the case $u_0 > u_1$, there is not elementary wave solution to the Riemann problem. One can try to substitute the SDW solution (see [4])

$$(\rho,u)(x,t) = \begin{cases} (\rho_0, u_0 + bt), & x < c(t) - \varepsilon t \\ (\rho_\varepsilon(t), u_\varepsilon(t)), & c(t) - \varepsilon t < x < c(t) + \varepsilon t \\ (\rho_1, u_1 + bt), & x > c(t) + \varepsilon t \end{cases} \tag{3}$$

in both equations of the system. Note that here $c(t)$ could not be const t, contrary to the case $b = 0$, since values on the both sides of a SDW are non-constant. The classical solution in the case $u_0 \le u_1$ satisfies all the usual admissibility criteria (entropy inequalities). As an admissibility criteria for SDWs we will use the overcompressibility condition.