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Relative entropy and a weak–strong uniqueness principle for the compressible Navier–Stokes equations on moving domains

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ABSTRACT

We establish a relative entropy inequality for the compressible Navier–Stokes equations, posed on domains with time-dependent moving boundaries. Using the relative entropy, a weak–strong uniqueness result is shown for the class of finite energy weak solutions.

1. Introduction

The compressible Navier–Stokes equations comprise a well-known model for the evolution of linearly viscous fluids. Whereas typically these PDEs are posed on fixed spatial domains, there is a large interest in the case of moving domains, sometimes referred to as non-cylindrical domains. In this paper, we assume the time-dependent boundary behavior is given, as opposed to free-boundary problems in which case the domain itself is unknown. Applications of Navier–Stokes on moving domains can be found in hemodynamics [1], respiratory systems [2], amongst others.

The objective of this paper is to establish a relative entropy inequality for compressible Navier–Stokes on moving domains. Relative entropies are functionals that measure a sort of distance between two solutions in a given function space. These functionals are often used to compare a weak solution with a (possibly hypothetical) strong or classical solution. This is particularly relevant for the case of Navier–Stokes equations, in which typically only weak solutions are known to exist. The principle of weak–strong uniqueness establishes that a weak and strong solution coincide, provided they both exist and have the same initial data. This principle can be deduced from the relative entropy inequality. A relative entropy and weak–strong result in the context of compressible Navier–Stokes on fixed domains are established in [3]. For similar results on the

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Navier–Stokes–Smoluchowski system, see [4]. Relative entropies in the context of hydrodynamic limits can be found in [5–7], amongst others.

The existence of weak solutions for incompressible Navier–Stokes on moving domains goes back to Sather [8]. In the case of compressible Navier–Stokes, weak solutions are established using penalization methods for no-slip boundaries in [9], and for slip boundaries in [10]. See also [11] for existence of weak solutions for a Navier–Stokes–Smoluchowski system on moving domains using penalization methods. The low Mach number limit for Navier–Stokes in a similar context is explored in [12].

With an evolving spatial domain, we can no longer consider Bochner-type functions mapping the time variable into a *fixed* function space. In order to make precise the functional setting, we choose function spaces similar to those defined in [13]. In this case, functions are defined via global extension by zero outside the moving fluid domain. These types of results are extended in [14,15], where so-called evolving spaces are defined via a pushforward/pullback map to associate function spaces at any time $t > 0$ with reference spaces at the initial time $t = 0$. For similar results on PDE with moving spatial domains, see [16,17], and the references therein.

The rest of the paper is outlined as follows. In Section 2, we discuss the governing equations and assumptions used in the sequel. Also in this section, we establish the functional framework needed to deal with the moving domain. In Section 3, we state the relevant existence result for weak solutions. The necessary energy inequality is also established. Finally in Section 4, we derive the relative entropy inequality, and deduce a weak–strong uniqueness result.

2. Preliminaries

The compressible Navier–Stokes equations consist of the conservation of mass, and conservation of momentum,

$$\begin{aligned} &\partial_t \rho + \operatorname{div}_x(\rho \mathbf{u}), \\ &\partial_t(\rho \mathbf{u}) + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\rho) = \operatorname{div}_x \mathbb{S} + \rho \mathbf{f}, \end{aligned} \tag{1}$$

where $\rho(t,x)$ is the fluid density, and $\mathbf{u}(t,x)$ is the fluid velocity. The external forcing is given by $\mathbf{f}(t,x)$, while the viscous stress tensor $\mathbb{S}$ is given through the Newtonian law,

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T \right) + \lambda \operatorname{div}_x \mathbf{u} \mathbb{I}, \tag{2}$$

where the (constant) viscosity coefficients satisfy

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu \geq 0.$$

We assume, for simplicity, that the pressure follows the γ-law

$$p(\rho) = a\rho^\gamma, \tag{3}$$

with constants $a > 0$ and $\gamma > \frac{3}{2}$.

The system (1) is posed on a moving spatial domain, which in turn is embedded in the "hold all" domain D. Indeed, for all $t \geq 0$, we assume that $\Omega_t \subset D \subset \mathbb{R}^3$ evolves according to a given "regular" velocity field $\mathbf{V} = \mathbf{V}(t, \mathbf{x})$, which is assumed to be divergence-free. The domain moves according to the flow generated by $\mathbf{V}$, given as the solution to

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t,\mathbf{x}) = \mathbf{V}(t, \mathbf{X}(t,\mathbf{x})), \quad t > 0, \\ \mathbf{X}(0,\mathbf{x}) = \mathbf{x}, \quad \mathbf{x} \in \Omega_0. \end{cases} \tag{4}$$

Indeed we have that $\Omega_t = \mathbf{X}(t, \Omega_0)$, where $\Omega_0 \subset D$ is a bounded initial domain.