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A degree-increasing [N to $N+1$] homotopy for Chebyshev and Fourier spectral methods

John P. Boyd

Department of Climate and Space Sciences and Engineering, University of Michigan, 2455 Hayward Avenue, Ann Arbor MI 48109-2143, United States

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ABSTRACT

Hitherto, as a tool for tracing all branches of nonlinear differential equations, resolution-increasing homotopy methods have been applied only to finite difference discretizations. However, spectral Galerkin algorithms typically match the error of fourth order differences with one-half to one-fifth the number of degrees of freedom N in one dimension, and a factor of eight to a hundred and twenty-five in three dimensions. Let $\vec{u}_N$ be the vector of spectral coefficients and $\vec{R}_N$ the vector of N Galerkin constraints. A common two-part procedure is to first find all roots of $\vec{R}_N(\vec{u}_N) = \vec{0}$ using resultants, Groebner basis methods or block matrix companion matrices. (These methods are slow and ill-conditioned, practical only for small N.) The second part is to then apply resolution-increasing continuation. Because the number of solutions is an exponential function of N, spectral methods are *exponentially* superior to finite differences in this context.

Unfortunately, $\vec{u}_N$ is all too often outside the domain of convergence of Newton's iteration when N is increased to $(N+1)$. We show that a good option is the artificial parameter homotopy $\vec{H}(\vec{u};\tau) \equiv \vec{R}_{N+1}(\vec{u}) - (1-\tau)\vec{R}_{N+1}(\vec{u}_N)$, $\tau \in [0,1]$. Marching in small steps in τ, we proceed smoothly from the N-term to the $N+1$-term approximations.

1. Introduction

When a nonlinear differential or integral equation is discretized with N degrees of freedom (grid point values or Fourier coefficients), the result is a set of N nonlinear equations. If the nonlinearity is a polynomial in the unknown $u(x)$, the discretization is a system of N-variate polynomials.

Unfortunately, such systems are often very challenging. Recently, however, reliable solvers for polynomial systems have become widely available. Most are very slow and therefore limited to small N. As noted by Allgower, Bates, Sommese and Wampler [1], though, small-N solutions are only the "opening act". The "continuation method" is the simple but powerful strategy of varying a parameter (either physical or

E-mail address: jpboyd@umich.edu.

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numerical) in small steps, using the converged solution for the previous parameter value, or the extrapolation of several previous solutions, as the first guess for the next. In principle, one can track all the physically-interesting branches as accurately as one pleases by combining continuation in the physical parameters with resolution-increasing continuation in the number of grid points or spectral coefficients N. Unfortunately, N is always an integer and there is no compelling reason for the N-term solution to lie within the convergence domain of a rootfinding iteration in the $(N+1)$-dimensional parameter space.

The proposed remedy is an artificial parameter homotopy. If the system of nonlinear algebraic equations is denoted by $\vec{R}_N$ and its solution by $\vec{u}_N$, a vector of Fourier or Chebyshev coefficients, then the "Newton Degree-Increasing Spectral Homotopy" [DISH] is the one-parameter family of nonlinear polynomial or transcendental equations

$$\vec{H}(\vec{u}) \equiv \vec{R}_{N+M}(\vec{u}) - (1-\tau)\vec{R}_{N+M}(\vec{u}_N), \quad \tau \in [0,1] \tag{1}$$

where $M \geq 1$ is a positive integer. We mostly choose $M = 1$, but there is no inherent restriction on its size. The first step is to make a vector of length $(N+M)$ whose first N elements are the spectral coefficients that solve the degree N system and the remaining M elements are zeros. When $\tau = 0$, the homotopy is $\vec{R}_{N+M}(\vec{u}) - (1-\tau)\vec{R}_{N+M}(\vec{u}_N)$ whose exact solution is $\vec{u}_N$. When $\tau = 1$, the homotopy reduces to our target system $\vec{H}(\vec{u}; \tau = 1) = \vec{R}_{N+M}(\vec{u})$. Thus, the homotopy system does indeed smoothly morph from the solution we know to the solution we seek. We can march from N to $N + M$ using steps in the parameter τ as small as we please.

This "Newton homotopy" was "commonly used" by 1968 [2]. Newton-DISH is novel only as an instance of the family of Degree-Increasing Spectral Homotopies.

Trajectories may collide if τ is real; the collision remedy [unneeded here] is to march from $\tau = 0$ to $\tau = 1$ through a semicircle or other user-chosen contour in the complex plane [3].

2. Example

The "Fifth-Degree Korteweg–de Vries" [FKdV] equation is

$$-\nu\, u_{XXXXX} + u_{XXX} + (u-c)u_X = 0 \quad \text{[FKdV Eq.]} \tag{2}$$

subject to the periodic boundary condition that $u(X) = u(X + 2\pi)$ [4,5]. The Fourier approximation is

$$u_N(X) = A\cos(X) + \sum_{n=2}^{N} a_n \cos(nX). \tag{3}$$

The solution branches are parameterized by "amplitude" A, the coefficient of $\cos(X)$. The phase speed c is an unknown. The coefficient-fixed parameterization excludes the trivial solution that all coefficients are zero. The residual function is the result of substituting the Fourier series into the differential equation

$$R(x; c, a_2, \ldots, a_N) = -\nu\, u_{N,XXXXX} + u_{N,XXX} + (u-c)u_{N,X}. \tag{4}$$

Galerkin's method minimizes the residual through the constraints that the first N spectral coefficients of the residual are zero,

$$R_j = \frac{1}{\pi}\int_0^{\pi} \sin(X)\, R(x; c, a_2, \ldots, a_N)\, dX = 0, \quad j = 1, 2, \ldots, N. \tag{5}$$

The elements of the spectral homotopy from $N = 2$ to $N = 3$ are

$$H_1 = \boxed{2A + 2\nu A + 2cA - Aa_2} - a_2 a_3 \tag{6}$$

$$H_2 = \boxed{64\nu a_2 + 16a_2 + 4ca_2 - A^2} - 2Aa_3 \tag{7}$$

$$H_3 = \{54 + 486\nu + 6c\}a_3 - \tau 3Aa_2. \tag{8}$$